ELEMENTS OF ANALYTICAL GEOMETRY.

MACMILLAN AND CO., LIMITED
LONDON · BOMBAY · CALCUTTA · MADRAS
MELBOURNE

THE MACMILLAN COMPANY
NEW YORK · BOSTON · CHICAGO
DALLAS · SAN FRANCISCO

THE MACMILLAN COMPANY
OF CANADA, LIMITED
TORONTO

ELEMENTS OF

ANALYTICAL GEOMETRY

BY

GEORGE A. GIBSON, M.A., LL.D.

PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF GLASGOW

AND

P. PINKERTON, M.A., D.Sc.

RECTOR OF THE HIGH SCHOOL OF GLASGOW

MACMILLAN AND CO., LIMITED

ST. MARTIN'S STREET, LONDON

1929

First Edition 1911.
Reprinted 1912, 1919, 1920, 1924, 1929.

PRINTED IN GREAT BRITAIN

TO

WILLIAM JACK, M.A., LL.D., D.Sc.

EMERITUS PROFESSOR OF MATHEMATICS IN THE

UNIVERSITY OF GLASGOW

FROM TWO FORMER STUDENTS

PREFACE.

In the preparation of this text-book on Analytical Geometry it has been our aim not merely to give an analytical treatment of curves of the second degree, but also to apply the methods of elementary algebra to the tracing of curves of higher degrees. Many of the curves usually classed as Higher Plane Curves and discussed in treatises on the Calculus are easily handled by elementary methods, and give the beginner a much better knowledge of the value of analysis than can be derived from a study of the conic sections alone. An elementary knowledge of the methods of curve tracing is in fact a necessary preliminary to any discussion of Higher Plane Curves that is based on Higher Algebra and the Infinitesimal Calculus, and seems to come properly within the scope of an introduction to Analytical Geometry.

It may be useful to indicate the general lines on which the book has been constructed and to state briefly the reasons for the order adopted.

Chapters I.-IX. treat of the straight line, the circle and some simple curves that can be readily sketched from their definitions without recourse to elaborate algebraical analysis. Graphical work is now so common in the early stages of every mathematical course that it is fair to assume that every reader has some previous acquaintance with the graphical interpretation of equations of a simple type.

The early chapters are therefore designed to make the student quite familiar with fundamental formulae, such as the Section, Distance and Gradient Formulae, which occur so frequently in all applications, and to train him in the geometrical interpretation of formulae and equations by applying them to familiar and easily drawn curves. Indeed, the analytical treatment of the straight line and circle is necessary, not so much for the geometrical results as for the acquisition of facility in the use and interpretation of formulae; only by such practice can the beginner learn to see the geometry behind the analysis. These chapters include a discussion of Harmonic Ranges and Pencils and of the usual theorems on the Circle, including Coaxal systems. The ninth chapter contains the equations of the Conchoid, the Cissoid and the Witch, with the usual applications to the trisection of an angle and the duplication of the cube; experience proves that these curves are of real interest and stimulate pupils to further study. A number of worked examples on loci and two sets of Miscellaneous Examples conclude this section.

Chapters X.-XVII. discuss the graphical representation of equations. The aim of these sections is to enable the student to sketch pretty rapidly the forms of the curves represented by algebraic equations that are not of very complicated types; his work on the equations of loci in the earlier parts of the book will have suggested the necessity of this study. Considerable stress is laid on the method of Successive Approximations, and we believe the method to be both so simple and so fruitful that no apology is needed for the space given to its discussion. In the course of this discussion we have felt obliged to treat some parts of elementary algebra that are often imperfectly grasped by the beginner, such as discriminants, turning values, repeated and infinite roots, and have been led by a simple and

natural process to a statement of the derivatives of the simpler algebraic functions. We hope that the revisal of work which is treated with more or less fullness in most text-books of algebra will be justified by the light that the geometric interpretation casts on somewhat abstract algebraic theorems as well as by the use to which these discussions are put in the graphing of equations. The chapters on the Solutions of Equations and Harder Curves will, we trust, be found to offer some interest to every type of student, even if for no other reason than as providing variety in algebraic teaching.

The rest of the book, Chapters XVIII.-XXIV., contains a fairly complete treatment of the Conic. Many properties of the curves are most easily handled by the methods of Euclidean Geometry, and we have not hesitated to adopt such methods when there was distinct advantage in doing so, with the result that we have been able to incorporate the essentials of the older treatises on Geometrical Conics. It is hard to justify the separation of geometrical and analytical conics; at any rate it has seemed to us that such separation is totally unwarranted, and is even mischievous in an elementary text-book. We have tried to include all the important properties of conics that are of an elementary character, and to group them into a comparatively small number of theorems, so that the student may not be burdened by being confronted with propositions that are of no special importance. The numerous Exercises that are given in every chapter provide ample practice, both on the geometrical and on the analytical aspects of the treatment, and will, we hope, be found useful in emphasizing the fact that, after all, the one subject of study is geometry, even though the methods are twofold. The simplicity introduced by the use of Joachimsthal's Section-Equation is, we think, sufficient warrant for the place assigned to it;

and the discussion of Systems of Conics in Chapter XXIV. seems to sum up so naturally the general principles that underlie the applications of analysis to geometry that we hope it will not be considered to be too severe for an elementary text-book. Comparatively little stress has been laid on the General Equation of the Second Degree; its importance in an elementary course does not seem to us to demand a fuller treatment than has been given to it.

Professor Chrystal's text-books on Algebra are so fundamental in their character that it is impossible to write on any branch of algebra without showing traces of their influence, but we desire to make special acknowledgment of the great help we have derived from Chapter 25 of his *Introduction to Algebra*. Much of our work is little more than a restatement of the ideas there laid down. Again, in Chapter XVII. we have tried to give an elementary account of some of the more important methods developed with so much skill in Frost's treatise on *Curve Tracing* —a book which is now out of print.

A word may be said on the position assigned to Freedom Equations—a terminology that is, we believe, due to Professor Chrystal. From some points of view, for example in its bearing on Dynamics, the representation of a curve by freedom equations is quite as natural, and is much more useful than the representation by a constraint equation. But apart from such applications, the value of the specification of a point on a curve in terms of a single parameter has been always recognized in works on Analytical Geometry in the case of the Conic Sections, while the hole theory of Unicursal Curves is simply that of one rm of Freedom Equations. It seems proper therefore to troduce the student at the outset to this alternative method of representing the equation of a curve; the theory s not difficult and the gain in facility of graphical repre-

sentation is great. Elimination is often a difficult and tedious process, and may in many cases be dispensed with, so far as the representation of a curve is concerned, by making use of Freedom Equations.

The Answers to all the Examples have been worked out by Mr. A. M. Williams, M.A., B.Sc., who has also read the whole book in proof. Mr. Peter Ramsay, M.A., B.Sc., has subjected the Examples to a searching revision, and has independently verified the solutions. To both of these gentlemen we offer our hearty thanks for the extreme care and thoroughness with which they have carried out their laborious task. In many details the book owes much to the experience of Sir Richard Gregory, and we thank him sincerely for his helpful advice. We would also gratefully acknowledge the excellence of the work done by Messrs. MacLehose.

GEORGE A. GIBSON.
P. PINKERTON.

10 The University,
Glasgow, *December*, 1910.

CONTENTS.

CHAPTER I.

STEPS. POSITION-RATIO. SECTION-FORMULAE.

CHAPTER II.

RECTANGULAR AXES. COORDINATES. DISTANCE-FORMULA. SECTION-FORMULAE. LINEAR EQUATION.

CHAPTER III.

GRADIENT OF A STRAIGHT LINE. OBLIQUE AXES. POLAR COORDINATES. AREAS.

CHAPTER IV.

REPRESENTATION OF GEOMETRICAL LOCI BY ANALYTICAL EQUATIONS. THE STRAIGHT LINE.

CHAPTER V.

THE STRAIGHT LINE (*Continued*).

CHAPTER VI.

PAIRS OF STRAIGHT LINES. HARMONIC RANGES AND PENCILS. CHANGE OF ORIGIN AND AXES.

CHAPTER VII.

THE CIRCLE.

CHAPTER VIII.

COAXAL CIRCLES. POLE AND POLAR.

CHAPTER IX.

CONCHOID. CISSOID. WITCH. PARABOLA. ELLIPSE. HYPERBOLA.

CHAPTER X.

THE CONVERSE PROBLEM. GRAPHS OF EQUATIONS. POLYNOMIALS.

CHAPTER XI.

ROUGH GRAPHS OF RATIONAL FRACTIONS. FREEDOM EQUATIONS.

CHAPTER XII.

IRRATIONAL FUNCTIONS.

CHAPTER XIII.

SUCCESSIVE APPROXIMATIONS.

CHAPTER XIV.

DERIVATIVES OF POLYNOMIALS. MAXIMA AND MINIMA.

CHAPTER XV.

APPROXIMATE SOLUTION OF EQUATIONS.

CHAPTER XVI.

ASYMPTOTES.

CHAPTER XVII.

HARDER CURVES.

CHAPTER XVIII.

CANONICAL EQUATIONS OF THE CONIC SECTIONS.

CHAPTER XIX.

THE ELLIPSE AS THE ORTHOGONAL PROJECTION OF A CIRCLE.

CHAPTER XX.

GEOMETRICAL DISCUSSION OF SECANTS, TANGENTS AND NORMALS OF A CONIC.

CHAPTER XXI.

ANALYTICAL DISCUSSION OF SECANTS, TANGENTS AND NORMALS TO CONICS.

CHAPTER XXII

POLE AND POLAR

CHAPTER XXIII

DIAMETERS OF CONICS

CHAPTER XXIV.

GENERAL THEOREMS ON CONICS CONFOCAL CONICS CURVATURE

ELEMENTS OF ANALYTICAL GEOMETRY

CHAPTER I.

STEPS. POSITION-RATIO. SECTION-FORMULAE.

1. Positive and Negative Measure. Consider the line AB in Fig. 1, divided internally at P and externally at Q. We see that

$$AP+PB=AB, \quad \text{.........................(1)}$$

for

$$7+3=10;$$

but

$$AQ-QB=AB, \quad \text{.........................(2)}$$

for

$$14-4=10.$$

So long as P is between A and B, $AP+PB=AB$. Let P move up to and coincide with B; even now $AP+PB=AB$, for $PB=0$. Let P move *through* B to Q; then PB diminishes to zero, when P is at B, and appears again on the other side of B, after $PB=0$. Following the

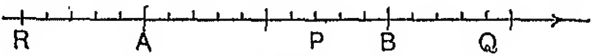


Fig. 1.

practice in Algebra, we could *measure* QB by (-4). Now $14+(-4)=10$; so that measuring QB (i) according to its length, by 4; (ii) according to the side of B on which it lies, by prefixing the sign $-$, we could write

$$AQ+QB=AB. \quad \text{.........................(3)}$$

If P had moved to the left of A, say to R, then AP would pass through zero to AR, while PB would grow steadily to RB. We could then denote AR by (-5), putting 5 for its length and prefixing the sign $-$, to make plain that AR lies on the side of A different from B. Now $(-5)+15=10$, so that we could write

$$AR+RB=AB. \qquad\qquad (3)$$

Starting from $AP+PB=AB$, it would be quite intelligible to read

$$AP+PB=AB,$$
$$AQ+QB=AB,$$
$$AR+RB=AB.$$

This may be summed up as follows:

Rule. *If A, B, P are any three points on a straight line,*

$$AP+PB=AB.$$

And the meaning of AP, PB, AB could be given thus:

On the line mark an arrow-head; if AP (or PB or AB), *from A to P*, follows the direction of the arrow-head, AP means the length of AP with the $+$ sign prefixed; if AP (or PB or AB), *from A to P*, follows the direction opposite to that of the arrow-head, AP means the length of AP with the $-$ sign prefixed.

In Fig. 1, $AP=(+7)$, $PB=(+3)$, $AB=(+10)$:

$$(+7)+(+3)=(+10),$$
$$\therefore\ AP+PB=AB.$$
$$AQ=(+14),\quad QB=(-4),\quad AB=(+10):$$
$$(+14)+(-4)=(+10),$$
$$\therefore\ AQ+QB=AB.$$
$$AR=(-5),\quad RB=(+15),\quad AB=(+10):$$
$$(-5)+(15)=(+10),$$
$$\therefore\ AR+RB=AB.$$

The rule $AP+PB=AB$ is a *general* rule; it enables us to be sure that a proposition, which depends on its use, is

true, whether P lies between A and B or not, provided that, as we say, we attend to the convention of sign regarding AP, PB, AB.

2. Origin and Axis of Abscissae. Let O be a fixed point on a line $X'OX$ (Fig. 2). Let U be another point on the line on the same side of O as X, and let the length of OU be one unit. The position of any point P on the line depends on *two* things, (1) the length of OP according to the scale $OU=1$, (2) the side of O on which P lies, whether the X-side of O or the X'-side. The length of OP, according to the scale $OU=1$, is specified by an arithmetical number, say 2·2. The side of O on which P lies is specified by prefixing to 2·2 the algebraic sign + or −; the sign + being prefixed if P lies on the same side of O as X, the sign − being prefixed if P lies on the same side of O as X'.

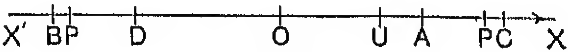


FIG. 2.

The number with the proper sign prefixed is called the **abscissa** of P with respect to the origin O, so that any abscissa can be entirely represented by an algebraic symbol, x, for example; since x, in Algebra, may stand for any arithmetical number with the sign + or − prefixed. Such a line as $X'OX$ is called an **axis of abscissae**, or simply an **axis**. OX is called the **positive direction** and OX' the **negative direction** of the axis. The positive direction may be indicated by an arrow-head. In Fig. 2 the abscissa of A is $+1·5$, or simply 1·5; the abscissae of B, C, D are $-2·4$, $2·4$, $-1·5$ respectively.

The abscissa of A is often denoted by OA; in this sense OA has a *double* significance, for it signifies both the magnitude and the sign of the abscissa of A. The measure of the length of OA, according to the scale $OU=1$, gives the *magnitude* of the abscissa; the *order of the letters* is the equivalent of the *sign* of the abscissa. OA, like x in Algebra, entirely represents the abscissa. Similarly we write

$$OB=-2·4,\ OC=2·4,\ OD=-1·5.$$

The measure of the *length* of OB is 2·4; the order of the letters, *from* O *to* B, signifies a motion in the negative direction of the axis; hence OB is entirely represented by $-2·4$.

If $X'OX$ is an axis, and P any point on it, we usually denote OP by x.

Ex. 1. Draw an axis of abscissae, choose an origin and scale [illegible], and mark the points whose abscissae are $2, -2, 1·7, -1·7$, [illegible]

Ex. 2. Plot with respect to an axis $X'OX$, scale unit 1 [illegible] points $x=2$, $x=-2·4$, $x=-3·2$, $x=2·8$.

3. Steps. Let A, B be two points on an axis, origin O, then AB can be measured (i), according to its length, by a number; (ii) by prefixing to this number the $+$ sign or the $-$ sign, according as the direction of AB, *from* A *to* B, is or is not the direction of the arrow-head on the axis, just as in § 2. When AB is measured in this way, AB is called **a step**. It is clear that

$$\text{step } AB = -\text{step } BA,$$

or, simply,

$$\mathbf{AB = -BA.}$$

Another important rule is

$$\mathbf{AB = OB - OA,}$$

for *every* position of the origin O.

First. Let OA, OB be both positive. In Fig. 3 (a).

$$AB = 5,\ OB = 7,\ OA = 2.$$

(a) (b)

FIG. 3.

$$AB = OB - OA.$$

Second. Let OA be positive, OB negative. In Fig. 3 (b).

$$AB = -5,\ OB = -3,\ OA = 2.$$

$$AB = OB - OA.$$

Third. Let OA, OB be both negative. In Fig. 3 (*c*),

$$AB=7,\ OB=-3,\ OA=-10.$$

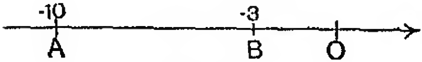


FIG. 3 (*c*).

$$AB=OB-OA.$$

Note also that

$$\mathbf{AO}+\mathbf{OB}=\mathbf{AB},$$

$$\mathbf{AO}+\mathbf{OB}+\mathbf{BA}=0$$

for every position of the origin.

Ex. 1. If the abscissae of A, B, points on an axis, origin O, have the following values, find the measure of the step AB:

(i) 3, 5; (ii) -4, 2; (iii) 3, -2; (iv) -1, -4.

Ex. 2. If A, B, C, D, points on an axis, have abscissae -3, 4, -8, -1 respectively, prove that $AB=CD$.

Ex. 3. If M is the middle point of AB, where A, B are points on an axis having abscissae (i) 4, 6; (ii) 4, -6; (iii) -4, 6; (iv) -4, -6; (v) a, b, find the abscissa of M in each case.

4. **Position-Ratio.** If A, B, P are three points on an axis, $\dfrac{AP}{PB}$ is called the **position-ratio** of P with respect to A, B. (Note that AP, PB are *steps*.) For example, in Fig. 4,

FIG. 4.

$$AP=OP-OA=(-1)-(-4)=3,$$

$$PB=OB-OP=1-(-1)=2,$$

$$\frac{AP}{PB}=\frac{3}{2}.$$

$$AQ=OQ-OA=11-(-4)=15,$$

$$QB=OB-OQ=1-11=-10,$$

$$\frac{AQ}{QB}=\frac{15}{-10}=-\frac{3}{2}.$$

Ex. 1. If A, B, P have abscissae

(i) 2, 7, 5; (ii) 2, 7, 10; (iii) -2, 7, 5; (iv) 2, -7, 5;
(v) -2, -7, -5,

find AP/PB in each case.

Ex. 2. If A, B, P, Q, four points on an axis, have abscissae -1, 2, 1, 5 respectively, prove that $AP/PB = -AQ/QB$.

5. First Section-Formula. If A has the abscissa 4 and B the abscissa 6, where A, B are points on an axis, and if AB is divided at P so that $AP/PB = 3/2$, then we could plot the points A, B on an axis, on which a scale-unit had been chosen, construct or mark the point P and read off the abscissa of P. This can be done whatever be the abscissae of A, B and the position-ratio AP/PB. Hence there must be a rule for calculating the abscissa of P in terms of the abscissa of A, the abscissa of B and the position-ratio AP/PB.

Rule. *Let $X'OX$ be an x-axis. Let the abscissae of two points A, B be x_1, x_2 respectively, let P be any point on the axis and $\frac{AP}{PB}=\frac{m}{n}$; then the abscissa x of P is found from the equation*

$$\mathrm{x}=\frac{\mathrm{mx}_2+\mathrm{nx}_1}{\mathrm{m}+\mathrm{n}}.$$

Proof. See Fig. 5.

FIG. 5.

$$AP = OP - OA = x - x_1,$$
$$PB = OB - OP = x_2 - x;$$
$$\therefore \frac{AP}{PB}=\frac{x-x_1}{x_2-x}.$$

But
$$\frac{AP}{PB}=\frac{m}{n};$$
$$\therefore \frac{x-x_1}{x_2-x}=\frac{m}{n};$$

$$\therefore\ nx - nx_1 = mx_2 - mx;$$
$$\therefore\ (m+n)x = mx_2 + nx_1;$$
$$\therefore\ x = \frac{mx_2 + nx_1}{m+n}.$$

Since only the *general* rule of § 3 has been used in the proof, the rule holds whether x_1, x_2 be positive or negative, and whether the position-ratio $\frac{m}{n}$ be positive or negative. If P lies within AB, then $\frac{m}{n}$ is positive; if P lies without AB, $\frac{m}{n}$ is negative. If then $\frac{p}{q}$ denote the *numerical* value of the position-ratio, we have the double rule

$$x = \frac{px_2 + qx_1}{p+q}, \text{ for internal section,}$$
$$x = \frac{px_2 - qx_1}{p-q}, \text{ for external section.}$$

COR. If x_1, x_2 are the abscissae of A, B and x the abscissa of the middle point, M, of AB, then $x = \frac{x_1 + x_2}{2}$. For we may put $m=1$, $n=1$.

EXERCISES I.

1. Find the abscissa of the middle point of the join of the points 3, 5.
("The point 3" is a contraction for "the point whose abscissa is 3.")

2. Find the abscissa of the middle point of the join of the points (i) -4, 2; (ii) -3, 5; (iii) -4, -2; (iv) 3, -5; (v) -3, -5.

3. Find the abscissae of the points of trisection of the join of (i) the points 2, 7; (ii) the points -4, 5; (iii) the points -1, -4.

4. A, B are the points 1, 5. AB is divided internally and externally at P, Q in the ratio 2/3; find the abscissae of P and Q, and calculate PQ.

5. A, B, C are the points -2, 3, 4; calculate AC/CB.

6. A, B have abscissae 2, 4, and AB is produced its own length through B to C; calculate AC/CB and the abscissa of C.

7. A, B have abscissae 2, 4, and AB is produced its own length through A to D; calculate AD/DB and the abscissa of D.

8. P divides the join of the points -3, 4 so that $AP/PB=-2$; find the abscissa of P.

9. A, B have abscissae a, b respectively, and AB is divided at P so that $AP/PB=(a-2b)/(2a-b)$; find the abscissa of P.

10. If $AP/PB=m/n$, prove that $AP/AB=m/(m+n)$.

11. If $AP/PB=\lambda$, establish the formula $x=\dfrac{x_1+\lambda x_2}{1+\lambda}$.

12. If $AP/AB=t$, establish the formula $x=x_1+t(x_2-x_1)$.

13. If A, B, M, P, Q are points on an axis such that M is the middle point of AB and $AP/PB=-AQ/QB$, prove that $MP\,.\,MQ=MA^2$. Prove also that $\dfrac{1}{AP}+\dfrac{1}{AQ}=\dfrac{2}{AB}$.

14. A, B have abscissae x_1, x_2; and P and Q divide AB internally and externally in the same ratio. If $PQ=d$, find the coordinates of P and Q.

15. A, B have abscissae x_1, x_2; and P and Q divide AB internally and externally in the ratio m/n. Calculate PQ in terms of x_1, x_2, m, n.

16. A, B have abscissae x_1, x_2; and the position-ratios of P and Q with respect to A, B are m, n respectively. Calculate PQ in terms of x_1, x_2, m, n.

6. Uniform Velocity. Suppose a point to move on an axis $X'OX$, unit 1 in., and let the following table be descriptive of the motion:

Position of point -	A	B	C	D	E
x=abscissa of point	-7	-4	2	11	20
t=time - - -	1	2	4	7	10

where the time, t, denotes the moment, reckoned in seconds from a certain zero, when the point is at A, B, C, etc.

Then the point moves from -7 to -4 in 1 second, *i.e.* moves $+3$ in. per second on an average, between A and B.

Also the point moves from -4 to 2 in 2 seconds, *i.e.* moves $+6$ in. in 2 seconds or $+3$ in. per second on an average, between B and C.

Similarly it moves between any two of the specified points at an average rate of $+3$ in. per second. The sign $+$ signifies that the motion is in the direction from X' to X.

If P, Q be *any* two points on an axis $X'OX$, unit 1 in., whose abscissae are x_1, x_2 respectively; if t_1, t_2 denote the times, in seconds, when a point moving on the axis is at P, Q respectively, then $\frac{PQ}{t_2-t_1}$ or $\frac{x_2-x_1}{t_2-t_1}$ (for $PQ=OQ-OP=x_2-x_1$) is the average velocity of the point between P and Q. If $\frac{PQ}{t_2-t_1}$ or $\frac{x_2-x_1}{t_2-t_1}$ is constant and $=v$, say, the point is said to have a uniform velocity v. If v is positive the motion is in the direction from X' to X; if v is negative, from X to X'.

The equation of the uniform velocity, described in the above table, is

$$x=-10+3t. \qquad \text{.........(1)}$$

For, put	$x=-7$	in equation (1).	Then	$-7=-10+3t$	or $t=1$.
„	$x=-4$	„ „	„	$-4=-10+3t$	or $t=2$.
„	$x=2$	„ „	„	$2=-10+3t$	or $t=4$.
„	$x=11$	„ „	„	$11=-10+3t$	or $t=7$.
„	$x=20$	„ „	„	$20=-10+3t$	or $t=10$.

Also, the velocity is uniform. For let $x=x_1$ and $t=t_1$ satisfy equation (1), and let $x=x_2$ and $t=t_2$ also satisfy the equation.

Then
$$x_1=-10+3t_1, \qquad \text{.........(2)}$$
$$x_2=-10+3t_2. \qquad \text{.........(3)}$$

From (3) subtract (2),
$$x_2-x_1=3(t_2-t_1);$$
$$\therefore\ \frac{x_2-x_1}{t_2-t_1}=3.$$

But $\frac{x_2-x_1}{t_2-t_1}$ is the average velocity between the points specified by x_1, x_2, and it is constant and equal to 3, whatever the points are; therefore the motion specified by equation (1) is that of a point moving on the axis with uniform velocity $+3$, *i.e.* moving in the direction from X' to X at the uniform speed of 3 inches per second.

Rule. If the motion of a particle on the axis $X'OX$, unit 1 in., is given by the equation $\mathbf{x=a+bt}$, t being reckoned in seconds, then the particle is moving on the axis with a uniform velocity of b inches per second, and the time is reckoned from the moment when the abscissa of the particle is a.

Ex. The motion of a point on the axis $X'OX$, unit 1 in., is given by the equation $x=3-2t$, when t is reckoned in seconds; calculate (1) when the point is at the origin; (2) where the point is initially; (3) where the point is, 2 seconds after zero-time; (4) where the poi

is, 2 seconds before zero-time; (5) the velocity of the point; (6) when the point has abscissa 5.

(1) Put $x=0$, $0=3-2t$, $t=1\frac{1}{2}$. The point is at the origin $1\frac{1}{2}$ secs. *after* zero-time.

(2) Put $t=0$, $x=3$. The point has abscissa $+3$.

(3) Put $t=+2$, $x=3-4=-1$. The point has abscissa -1.

(4) Put $t=-2$, $x=3+4=7$. " " " 7.

(5) We write $x_1=3-2t_1$, $x_2=3-2t_2$, whence $x_2-x_1=-2(t_2-t_1)$ or $\frac{x_2-x_1}{t_2-t_1}=-2$. The velocity is -2, *i.e.* the point moves in the direction from X to X' at the uniform rate of 2 inches per second.

(6) Put $x=5$, $5=3-2t$, $t=-1$. 1 sec. before zero-time.

7. Second Section-Formula. *If A, B, P are three points on the axis $X'OX$, if the abscissae of A, B are x_1, x_2 respectively and if $\frac{AP}{AB}=t$, then x, the abscissa of P, is found from the equation*

$$\mathbf{x}=\mathbf{x}_1+(\mathbf{x}_2-\mathbf{x}_1)\mathbf{t}.$$

Proof. See Fig. 5, p. 6.

$$AP=OP-OA=x-x_1,$$

$$AB=OB-OA=x_2-x_1.$$

$$\text{But } \frac{AP}{AB}=t;$$

$$\therefore \frac{x-x_1}{x_2-x_1}=t;$$

$$\therefore x-x_1=(x_2-x_1)t;$$

$$\therefore x=x_1+(x_2-x_1)t.$$

COR. The abscissa of the mid-point of the join of x_1, x_2 is $\frac{x_1+x_2}{2}$. (Put $t=\frac{1}{2}$.)

EXERCISES II.

1. The motion of a point on $X'OX$, unit 1 inch, is specified by the equation $x=3+4t$, t being reckoned in seconds; find (1) the velocity of the point at every instant; (2) the position of the point at zero-time;

(3) the position of the point 3 seconds after and 3 seconds before zero-time; (4) when, reckoned from zero-time, the point has abscissae 1, −1.

2. Find an equation to specify the motion of a point on the axis $X'OX$, unit 1 ft., if the point has a uniform velocity of +2 ft. per sec., and if the point has abscissa 1 at zero-time. (Time reckoned in seconds.)

3. Find an equation to represent the motion of a point on $X'OX$, unit 1 foot, if it has a uniform velocity of −3 feet per sec., and its abscissa is 2 at zero-time. (Time reckoned in seconds.)

4. At zero-time, a particle moving with uniform velocity on $X'OX$ has abscissa x_1; one second later it has abscissa x_2. Prove that, at time t seconds, it has the abscissa x where

$$x=x_1+(x_2-x_1)t.$$

5. Use the formula $x=x_1+(x_2-x_1)t$ to answer Exs. 1-3, p. 7.

6. A, B have abscissae 2, −3; and P and Q divide AB so that $AP/AB=3/5$ and $AQ/AB=-2/5$. Calculate the abscissae of P and Q and the measure of the step PQ.

7. A, B, C have abscissae 2, −3, −5; calculate AC/AB.

8. A, B have abscissae −2, −3. Find the distance between the points which divide AB internally and externally in the ratio 3 : 4.

9. P, Q divide AB internally and externally in the same ratio. Find the abscissa of Q in terms of x_1, x_2, t, if x_1, x_2 are the abscissae of A, B and $AP/AB=t$.

10. If A, B, C, D be any four points on an axis, prove that

$$BC \,.\, AD+CA \,.\, BD+AB \,.\, CD=0.$$

11. If A, B, C be any three points on an axis and O the middle point of AB, prove that $AC^2-CB^2=2AB \,.\, OC$.

12. If A, B, C, P be any four points on an axis and $AP/PB=n/m$, prove that

$$m \,.\, AC^2+n \,.\, BC^2=m \,.\, AP^2+n \,.\, BP^2+(m+n)CP^2.$$

CHAPTER II.

RECTANGULAR AXES. COORDINATES. DISTANCE FORMULA. SECTION FORMULAE. LINEAR EQUATION

8. Rectangular Axes. Coordinates of a Point. In Fig. 6, let $X'OX$ be an axis of abscissas. Let $Y'OY$ be drawn perpendicular to $X'OX$. Then $Y'OY$ may be used as a second axis, and OY taken as the positive direction of the

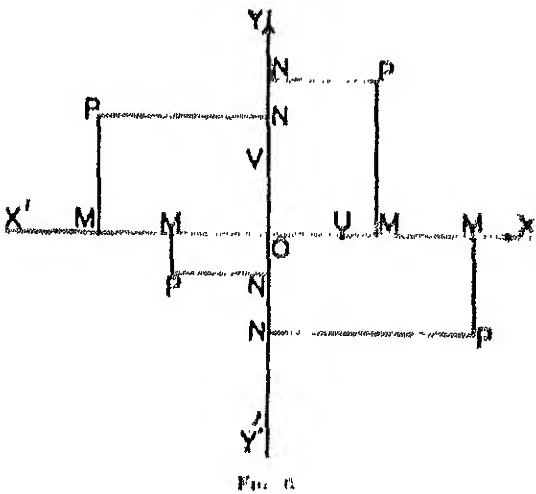


Fig. 6.

axis. Let OU, OV be the scale units of the axes $X'OX$, $Y'OY$ respectively (in Fig. 6 the units are equal), let P

lie on the same side of O as X, and let V lie on the same side of O as Y.

Let P be *any* point in the plane of the axes and let M, N be the projections of P on $X'OX$, $Y'OY$ respectively. Since M lies on the axis $X'OX$, called **the x-axis**, the position of M is specified by OM. Similarly the position of N on $Y'OY$, called **the y-axis**, is specified by ON. Let $OM=x$ and $ON=y$; then the position of P is specified when x and y are known, and conversely x and y are known when the position of P is determined.

OM or x is called **the x-coordinate or abscissa** of P; ON or y is called **the y-coordinate or ordinate** of P; x and y are called **the coordinates** of P; $X'OX$ and $Y'OY$ are called **the coordinate axes**, and are **rectangular axes**; P is called **the point (x, y)**.

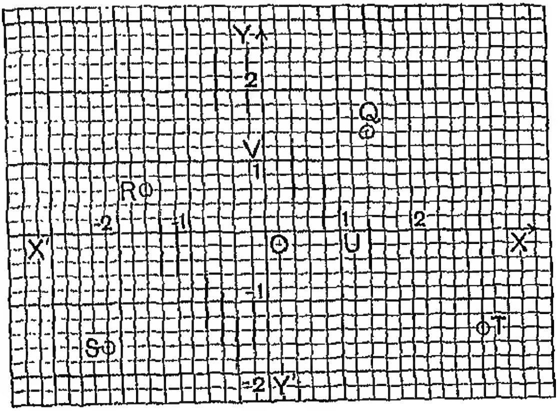


Fig. 7.

Clearly MP may be used instead of ON; for their lengths are the same and the direction from M to P is the same as the direction from O to N, the positive direction being that of OY, the negative direction that of OY'. Hence if P is the point (x, y), $OM=x$, $MP=y$. In Fig. 7, Q is the point $(1{\cdot}4, 1{\cdot}4)$, R is the point $(-1{\cdot}6, 0{\cdot}6)$, S the point $(-2{\cdot}2, -1{\cdot}6)$, T is the point $(3, -1{\cdot}4)$.

The axes $X'OX$, $Y'OY$ and the coordinates x, y are [illegible] called **Cartesian** axes and coordinates.

Ex. 1. Draw rectangular axes $X'OX$, $Y'OY$; let the scale [illegible] each axis be one centimetre. Mark the positions of the points [illegible] $(2, -3)$, $(-2, 3)$, $(-2, -1)$, $(2{\cdot}4, -1{\cdot}8)$, $(-1{\cdot}2, -2{\cdot}2)$.

Ex. 2. Mark on another drawing of the axes of Ex. 1 the follo[illegible] pairs of points and calculate the distance between the points, [illegible] the calculation by measurement:

(1) $(1, 2)$ and $(4, 6)$; (2) $(-2, 2)$ and $(1, 6)$;

(3) $(8, -1{\cdot}5)$ and $(-4, -6{\cdot}5)$; (4) $(-2{\cdot}3, 3{\cdot}1)$ and $(1{\cdot}$[illegible]

9. Distance-Formula. Let P be the point (x_1, y_1) and [illegible] the point (x_2, y_2), referred to chosen rectangular axes; [illegible]

$$\mathbf{PQ} = \pm\sqrt{(\mathbf{x}_1 - \mathbf{x}_2)^2 + (\mathbf{y}_1 - \mathbf{y}_2)^2}.$$

Let M, N (Fig. 8) be the projections of P, Q on $X'OX$; let PR, a parallel to $X'OX$, meet NQ in R.

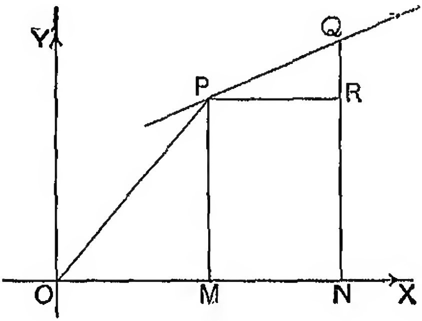


FIG. 8.

Then $$PR = MN = ON - OM = (x_2 - x_1);$$

$$\therefore\ PR^2 = (x_2 - x_1)^2.$$

Also $$RQ = NQ - NR = NQ - MP = (y_2 - y_1);$$

$$\therefore\ RQ^2 = (y_2 - y_1)^2.$$

But $$PQ^2 = PR^2 + RQ^2;$$

$$\therefore\ PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2;$$

$$\therefore \quad PQ = \pm\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

or

$$\pm\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}.$$

We have seen (§ 8) that a straight line parallel to either axis may also be used as an axis, its positive direction being that of OX or OY. When a line PQ is not parallel to either axis, the line may still be regarded as an axis, but its positive direction has no dependence on the positive direction of OX or OY; hence the ambiguity of sign in the distance-formula. In the meantime, let us agree that the positive direction of such an axis be the direction of motion of a point which travels along the line so that its abscissa

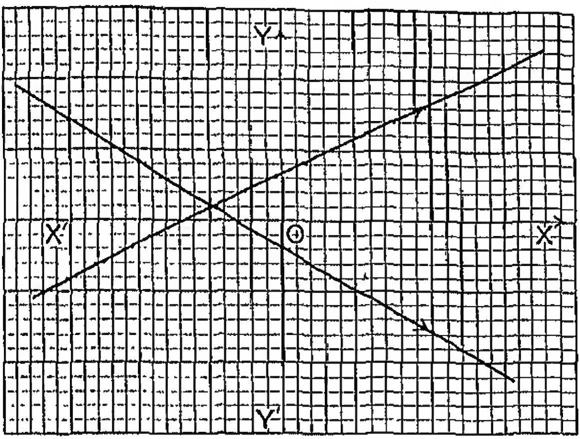


FIG. 9.

steadily increases. Thus, in Fig. 9, the positive directions of the lines are as indicated by arrow-heads. With this convention

$$PQ = (x_2-x_1)\sqrt{1+\left(\frac{y_2-y_1}{x_2-x_1}\right)^2}$$

in sign and magnitude, the positive value of the root being understood.

Note that $OP = \pm\sqrt{x_1^2+y_1^2}$. In sign and magnitude $OP = x_1\sqrt{1+y_1^2/x_1^2}$.

Ex. 1. Calculate, by the formula, the distance between the following pairs of points:

(1) (1, 2) and (4, 6); (2) (−2, 2) and (1, 0);
(3) (8, −1·5) and (−4, −6·5).

Ex. 2. A, B, P are three collinear points; calculate the sign and magnitude of AP/PB and AP/AB when A, B, P have the following coordinates respectively:

(1) (1, 2), (2, 2), (3, 2); (2) (2, −1), (2, 1), (2, 2);
(3) (1, 2), (3, 6), (2, 4); (4) (1, 2), (3, 6), (4, 8).

Ex. 3. Show that the points (2, 5), (5, 2), (6, 6) are the vertices of an isosceles triangle.

Ex. 4. Show that (2, −2), (5, 2), (−2, 1) are the vertices of a right-angled triangle, and find its area.

Ex. 5. One end of a line whose length is 13 is the point ( ), and the ordinate of the other end is 1. What are the possible values of its abscissa?

Ex. 6. Show that the following points lie on a circle whose centre is the point (3, 4) and whose radius is 5:

(8, 4), (7, 7), (6, 8), (0, 8), (−1, 1), (3, −1).

Ex. 7. Calculate the sides and diagonals of the quadrilateral whose vertices are (3, 2), (−1, 3), (0, 0), (4, 0); and test your results by measurement.

Ex. 8. If (x, y) is any point which lies on a circle, centre (2, 3), and radius 4, prove that $x^2+y^2-4x-6y-3=0$.

10. First Section-Formula. *If fixed points A, B have coordinates (x_1, y_1) and (x_2, y_2) respectively, and if a variable point P in the AB-axis have coordinates (x, y), then we may write*

$$\mathbf{x}=\frac{\mathbf{m}\mathbf{x}_2+\mathbf{n}\mathbf{x}_1}{\mathbf{m}+\mathbf{n}}; \quad \mathbf{y}=\frac{\mathbf{m}\mathbf{y}_2+\mathbf{n}\mathbf{y}_1}{\mathbf{m}+\mathbf{n}},$$

where $AP/PB=m/n$ $(\neq -1)$.

Proof. Let F, H, M (Fig. 10) be the projections of A, B, P respectively on $X'OX$;

let G, K, N be the projections of A, B, P respectively on $Y'OY$;

let AG meet MP in Q and HB in R;

let AF meet NP in S and KB in T.

Then $$\frac{AP}{PB}=\frac{AQ}{QR}=\frac{FM}{MH};$$

therefore $$\frac{m}{n}=\frac{OM-OF}{OH-OM}=\frac{x-x_1}{x_2-x},$$

so that $$nx-nx_1=mx_2-mx$$

or $$(m+n)x=mx_2+nx_1;$$

and therefore $$x=\frac{mx_2+nx_1}{m+n}.$$

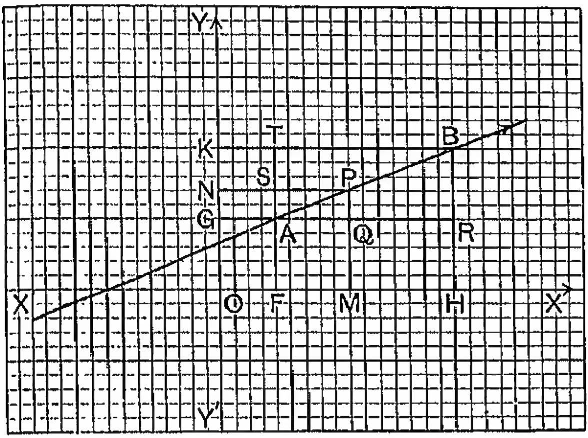


FIG. 10.

Similarly, $$\frac{AP}{PB}=\frac{AS}{ST}=\frac{GN}{NK}=\frac{ON-OG}{OK-ON},$$

so that $$\frac{m}{n}=\frac{y-y_1}{y_2-y};$$

and therefore $$y=\frac{my_2+ny_1}{m+n}.$$

COR. The coordinates of the middle point of AB are
$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

EXERCISES III.

1. Find the coordinates of the middle point of the join of (2, 3) and (4, 5).

$$x=\frac{x_1+x_2}{2}=\frac{2+4}{2}=3;\quad y=\frac{y_1+y_2}{2}=\frac{3+5}{2}=4.$$

The coordinates of the middle point are (3, 4).

2. Find the middle point of the join of $(-4, 7)$, $(2, -3)$.

3. Find the points of trisection of AB where A is the point (2, 3) and B is the point (4, 5).
Using the formulae, put $m=1$ and $n=2$, then $m=2$, $n=1$.

4. If A, B are the points $(-1, 4)$ and $(5, -2)$ respectively, find the coordinates of P (1) when $AP/PB=1$, (2) when $AP/PB=2$, (3) when $AP/PB=-2$, (4) when $AP/PB=-4/3$.

5. A, B are the points $(-2, 5)$, $(7, 1)$. Find the coordinates of P, Q which respectively divide AB internally and externally in the ratio 3/2. Calculate the length of PQ.

6. A, B are the points $(3, -5)$, $(-6, 2)$; and P divides AB so that $AP/PB=-2/3$. Calculate the lengths of AP and PB.

7. A, B are the points (11, 0) and $(-10, 0)$; and C is the point $(-5, 12)$. The internal bisector of angle C of triangle ABC meets AB in P. Calculate (1) AP/PB, (2) the abscissa of P, (3) the length of PC.

8. A, B, C are the points $(-13, 0)$, $(15, 0)$, $(-5, 15)$ respectively. The in- and ex-bisectors of the angle C of triangle ABC meet AB in P and Q respectively. Calculate the lengths of PQ, PC, QC.

9. A, B, C are the points $(2, 3)$, $(7, -5)$, $(-4, -8)$. Calculate the coordinates of the centroid of triangle ABC.

10. Prove that $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$ is the centroid of the triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3).

11. If A, B, C are the points $(5, 0)$, $(-5, 0)$, $(3, 6)$ and (x, y) is any point P in their plane, prove that

$$PA^2+PB^2+PC^2=GA^2+GB^2+GC^2+3PG^2,$$

where G is the point (1, 2).

12. If A, B, C are the points $(a, 0)$, $(-a, 0)$, (b, c), and P is any point on the x-axis such that $AP/PB=n/m$, prove that

$$m\,.\,AC^2+n\,.\,BC^2=m\,.\,AP^2+n\,.\,BP^2+(m+n)\,.\,CP^2.$$

13. If P, Q are the points $(a\cos\theta, b\sin\theta)$, $(-a\sin\theta, b\cos\theta)$, prove that $OP^2+OQ^2=a^2+b^2$, where O is the origin.

11. Component Velocities: Resultant Velocity. Can a moving body be travelling in two different directions at one and the same time? Can a balloon be said to be travelling forwards and upwards at the same time, or must we say that it is travelling in just one slanting direction at any time? If a man walks from the front towards the rear of a corridor train travelling west, is he moving both east and west at the same time or is he simply still travelling west? If a ring is rolled along a table, is a point on the ring going round and also going forward at the same time? If a little lamp were placed on the ring and the ring rolled along a table in a dark room, so that an observer did not know how the motion was produced, would he dream of saying that the lamp was going round and also going forward; would he not say simply that the lamp was moving sideways down to or up from the floor?

The two possible answers, yes and no, to these questions seem to be contradictory; but they are not contradictory. It is true enough to say that the balloon is moving forwards and upwards at the same time; it is equally true to say that it is going in one definite slanting direction at any moment.

To avoid confusion, however, we say that a body may have two (or more) **component** motions or displacements at one and the same time, or a single **resultant** motion or displacement at any one time. If a point is moving in the plane of the axes $X'OX$, $Y'OY$ we are free to consider its component motions in the directions of $X'OX$ and $Y'OY$ separately with the object of answering any question about the motion.

For example, let the curved line AB in Fig. 11 represent a telegraph wire suspended from A and B, and running alongside a railway line CD; the line AB is curved because of the sag in the wire. Let a train

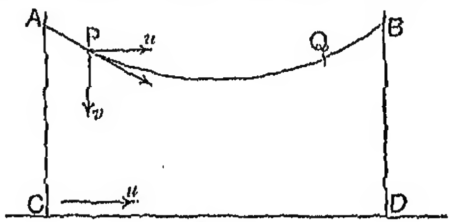


FIG. 11.

be supposed to move from C to D with uniform speed u, and let a passenger seated in the train watch the wire as he passes. Let us further suppose that a particularly energetic fly travels along the wire from A to B so as to be always opposite to the passenger. When the fly is at P its resultant velocity is in the direction of the tangent to the wire at P, which slopes downwards. Now, applying the notion of component velocities, think of the fly when at P as moving forward horizontally and downwards vertically at one and the same time

instead of moving in the direction of the tangent. Since the fly keeps opposite the passenger its horizontal motion must keep pace with that of the passenger; the fly therefore moves horizontally forwards with velocity u and vertically downwards with a velocity that we may call v. Now it is a matter of common observation that if two trains move side by side with the same velocity a passenger in one would think a passenger opposite to him in the other was not moving at all; they have no *relative* velocity. Hence the fly when at P must seem to the passenger to be falling vertically downwards; at Q, on the other hand, the fly would appear to be rising straight up.

The illustration shows how a body which is moving with a definite velocity may be regarded as having two or more component velocities of which the definite velocity is the resultant. The resultant velocity may be regarded as a constraint velocity, the component velocities as freedom velocities.

12. Parallelogram of Velocities. Let a moving point, when at O, the origin of the axes $X'OX$, $Y'OY$, have a component velocity of 2 inches per second in the direction $X'OX$ and a component velocity of 1 inch per second in the direction $Y'OY$, and let it move for a certain time, its component velocities remaining the same during that time (Fig. 12). At the end of half a second it will arrive at the

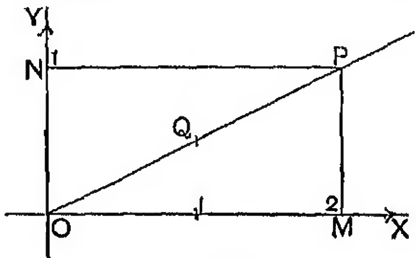


FIG. 12.

position $Q\ (1, \frac{1}{2})$; at the end of one second it will arrive at the position $P\ (2, 1)$; at the end of t seconds it will arrive at the position $(2t, t)$. All these positions are on the straight line OP or OP produced. Hence if OM, ON be cut off from the axes to represent the component velocities the diagonal OP of the rectangle $OMPN$ will represent the resultant velocity. If OX, OY are not at right angles a parallelogram $OMPN$ would replace the rectangle. Hence a uniform velocity in the plane of the axes may be replaced by two component uniform velocities in the directions of the axes.

13. Freedom Equations: $x=a+bt$, $y=c+dt$.

Any component uniform motion along the axis $X'OX$ is specified (§ 6) by the equation $x=a+bt$, a being the abscissa of the moving point at zero-time, and b the velocity.

Any component uniform motion along $Y'OY$ is specified by the equation $y=c+dt$, c being the ordinate of the moving point at zero-time, and d the velocity.

Hence if a point move with uniform velocity along any straight line in the plane of the axes $X'OX$, $Y'OY$, the motion is completely specified by the freedom equations

$$x=a+bt, \quad y=c+dt.$$

14. Second Section-Formula. *If fixed points A, B have coordinates (x_1, y_1), (x_2, y_2) respectively, and if a variable point P in the AB-axis have coordinates (x, y), then we may write*

$$\mathbf{x}=\mathbf{x}_1+(\mathbf{x}_2-\mathbf{x}_1)\mathbf{t}, \quad \mathbf{y}=\mathbf{y}_1+(\mathbf{y}_2-\mathbf{y}_1)\mathbf{t},$$

where $t=AP/AB$.

Proof. In Fig. 10,

$$\frac{AP}{AB}=\frac{AQ}{AR}=\frac{FM}{FH}=\frac{OM-OF}{OH-OF}.$$

But
$$\frac{AP}{AB}=t \text{ and } \frac{OM-OF}{OH-OF}=\frac{x-x_1}{x_2-x_1};$$

$$\therefore \quad \frac{x-x_1}{x_2-x_1}=t;$$

$$\therefore \quad x=x_1+(x_2-x_1)t.$$

Similarly,
$$y=y_1+(y_2-y_1)t.$$

COR. The coordinates of the middle point of the line joining the points (x_1, y_1) and (x_2, y_2) are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. [Put $t=1/2$.]

Ex. 1. If A, B are the points $(-2, 3)$ and $(5, -1)$ respectively, find the coordinates of P (1) when $AP/AB=1/3$, (2) when $AP/AB=-3/2$.

Ex. 2. If (x, y) are the coordinates of *any* point on the line joining $(2, 1)$ and $(5, 3)$, prove that $2x-3y=1$.

Put
$$x=x_1+(x_2-x_1)t=2+3t,$$
$$y=y_1+(y_2-y_1)t=1+2t.$$

Now t is the same number in both of these equations. From the first, $t=(x-2)/3$, from the second $t=(y-1)/2$. Hence

$$\frac{x-2}{3}=\frac{y-1}{2}, \quad i.e.\ 2x-3y=1.$$

Ex. 3. If (x, y) is any point on the line joining the points ([illegible] (1, −5), then $3x+2y+7=0$.

Ex. 4. If (x, y) is any point on the line joining the points [illegible] $(-4, -2/3)$, prove that $2x-3y+6=0$.

15. The equation $Ax+By+C=0$.

If (x, y) is any point on a given straight line, [illegible] $Ax+By+C=0$; where A, B, C are constants which [illegible] in specifying the line.

Let the line be specified by fixing two points on it, and let these points be (x_1, y_1) and (x_2, y_2).

Then (§ 14) we may write

$$x=x_1+(x_2-x_1)t,$$

$$y=y_1+(y_2-y_1)t,$$

or

$$x=a+bt, \quad \text{......} \quad (1)$$

$$y=c+dt, \quad \text{......} \quad (2)$$

where a, b, c, d are constants arising out of the specification of the line.

From (1), $\quad dx=ad+bdt,$

" (2), $\quad by=bc+bdt;$

subtract: $\quad dx-by=ad-bc,$

i.e. $\quad dx+(-b)y+(bc-ad)=0,$

which may be written

$$Ax+By+C=0.$$

An equation of the form $Ax+By+C=0$, where A, B, C do not depend on x, y, is called an equation of the first degree in x, y or a linear equation in x, y.

Hence we may enunciate the theorem of this section as follows:

The coordinates (x, y) of every point on a straight line satisfy an equation of the first degree in x, y.

Ex. 1. If (x, y) is any point on the straight line given by the points (2, 1) and (3, 3) on it, prove that $2x-y-3=0$. Verify the equation when (x, y) is (1) the point of bisection of the line joining the points (2, 1) and (3, 3); (2) either point of trisection of the line joining the points (2, 1) and (3, 3).

Ex. 2. Prove that the equation $2x+y=5$ is satisfied by the coordinates of every point in the line joining (2, 1) and (−1, 7). Calling these points A, B respectively, verify that the coordinates of P satisfy the equation

(1) when $AP/PB=1$; (2) when $AP/PB=-1/2$;
(3) when $AP/AB=-1/2$.

Ex. 3. A particle starts from the point (a, b) and travels so that its component velocities parallel to $X'OX$ and $Y'OY$ are the constants u, v; prove that the coordinates of its position at any time during the motion satisfy the equation $vx-uy=va-ub$. Why do $x=a+u$, $y=b+v$ satisfy the equation? (See §§ 6, 13.)

Ex. 4. Prove that the points (1, 3), (5, −1), (−2, 6) are collinear.
Let (x, y) be any point on the join of (1, 3), (5, −1); then $x+y=4$. But $(-2)+6=4$; ∴ etc.

Ex. 5. Prove that the four points (−2, 3), (2, 7), (4, 9), (−1, 4) are collinear.

16. The Equation of a Straight Line. It has been seen (§ 15) that the coordinates x, y of *any* point on a given straight line satisfy an equation of the first degree in x, y. This equation is called **the equation of the straight line.** For example, the coordinates of any point on the straight line passing through (2, 1) and (5, 3) satisfy the equation $2x-3y=1$ (§ 14, Ex. 2). $2x-3y=1$ is the equation of the straight line passing through (2, 1) and (5, 3); we also, for shortness, speak of "the straight line $2x-3y=1$" instead of "the straight line whose equation is $2x-3y=1$."

The equations

$$x=x_1+(x_2-x_1)t, \quad y=y_1+(y_2-y_1)t$$

or

$$x=a+bt, \qquad y=c+dt$$

are called **the freedom equations** of a straight line. Thus

$$x=2+3t, \quad y=1+2t$$

are freedom equations of the line whose **constraint equation** is

$$2x-3y=1.$$

The following examples will show how the specification of a given straight line is translated into an analytic equation; and, conversely, how a linear equation in x, y is represented by a straight line.

Ex. 1. Find the equation, referred to chosen or assigned axes and scale units,* of the straight line passing through the origin and the point (3, 5).

Let the axes and scale units be those of Fig. 13. Let A be the point (3, 5) and P *any* point (x, y) on the line OA, whose equation is required.

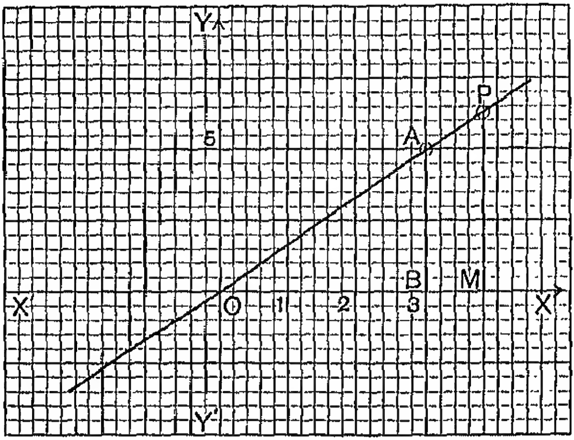


Fig. 13.

Let BA, MP be the ordinates of A, P.

Then Δs OAB, OPM are similar.

$$\therefore \frac{MP}{BA} = \frac{OM}{OB} \quad \text{.............................(1)}$$

This equation is true in sign as well as magnitude for every position of P, since MP and OM have always the same sign, and BA and OB are positive.

But, in sign and magnitude, $MP=y$, $BA=5$, $OM=x$, $OB=3$. Substituting these values in (1), we have

$$\frac{y}{5} = \frac{x}{3};$$

$$\therefore \; 5x = 3y.$$

Hence the coordinates of any point on the line satisfy the equation $5x=3y$,

i.e. $5x=3y$ is the equation of the line.

(Note that the equation is satisfied if $x=0$, $y=0$, and also if $x=3$, $y=5$.)

* This clause is usually left to be understood.

Ex. 2. Through the point (0, 2) is drawn a straight line parallel to the straight line passing through the points (0, 0) and (3, 5); to find the equation of the parallel.

Let the axes and scale-units be those of Fig. 14. Let P be any point (x, y) on the parallel, MP the ordinate of P, cutting the line joining (0, 0) and (3, 5) in Q; let C be the point (0, 2).

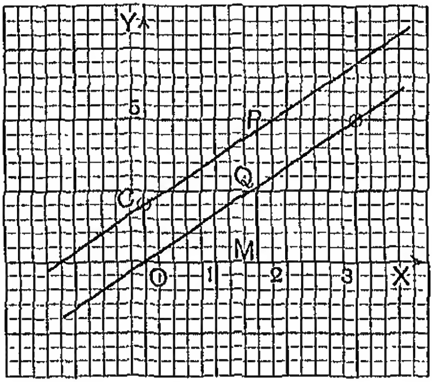


Fig. 14.

Then $MP = MQ + QP$, for all positions of M, Q, P (§ 3).

$$\therefore\ MP = MQ + OC. \quad \text{.................................(1)}$$

Now, by Ex. 1, the equation of OQ is $5x = 3y$,

$$\therefore\ 5OM = 3MQ,$$

$$\therefore\ MQ = \tfrac{5}{3} \cdot OM.$$

Substituting in (1), we have

$$MP = \tfrac{5}{3} \cdot OM + OC.$$

But $MP = y$, $OM = x$, $OC = 2$.

$$\therefore\ y = \tfrac{5}{3}x + 2,$$

$$\therefore\ 5x - 3y + 6 = 0,$$

i.e. the coordinates of any point on the parallel satisfy the equation $5x - 3y + 6 = 0$.

$\therefore\ 5x - 3y + 6 = 0$ is the equation of the parallel.

(Note that the equation is satisfied if $x = 0$, $y = 2$.)

Ex. 3. Find the equation of the straight line passing through the points $(2, -3)$ and $(4, -6)$.

Let (x, y) be any point on the line.

We have (§ 14) $x=x_1+(x_2-x_1)t,$

$$y=y_1+(y_2-y_1)t.$$

Therefore $x=2+(4-2)t=2+2t,$

$$y=-3+(-6+3)t=-3-3t,$$

so that $3x+2y=0$ is the required equation.

Ex. 4. Axes and scale-units being chosen, draw the straight line represented by the equation $2x+3y+5=0$.

$$\text{When } x=-4,\quad -8+3y+5=0\,;\quad \therefore\ y=1.$$

$$\text{When } x=5,\quad 10+3y+5=0\,;\quad \therefore\ y=-5.$$

Hence $(-4, 1)$, $(5, -5)$ are on the line required. Plot these two points and draw a straight line through them. The method is thus simply: choose any two convenient values of x, calculate from the equation the corresponding values of y, and then plot the two points. Care should be taken to select points not very close to each other, and it is often useful to plot three points as a test of the accuracy of the drawing.

EXERCISES IV.

1. The equation of the parallel to $Y'OY$ through the point $(-2, 0)$ is $x+2=0$.

2. The equation of the parallel to $Y'OY$ through the point $(3, 4)$ is $x=3$.

3. What straight lines are specified by the equations $x=1$, $x+1=0$, $y+2=0$, axes and scale-units being previously assigned?

4. What is the equation of the locus traced out by a point which starts from the position $(0, -3)$ and moves parallel to the x-axis?

5. The equation of the bisector of the angles XOY, $X'OY'$ is $x-y=0$.

6. If the scale-unit of the x-axis is one inch and the scale-unit of the y-axis half an inch, draw the line whose equation is $x-y=0$. By Example 5 the equation $x-y=0$ represents the bisector of the angle XOY; does the line you have drawn bisect the angle XOY? Show that the line you have drawn should have for its equation, *if the scale-units were the same for the two axes*, $x-2y=0$.*

7. The equation of the bisector of the angles YOX', $Y'OX$ is $x+y=0$.

8. The equation of the straight line joining the origin to the point $(2, 3)$ is $3x=2y$.

*This example shows how a diagram is distorted when the scale-unit of the x-axis is different from that of the y-axis. If a diagram is not to be distorted the two scale-units must be the same.

9. A is the point (3, 4) and O is the origin. The equation of OA is $4x=3y$.

10. The equation of the line joining the origin and $(-2, 3)$ is $3x+2y=0$.

11. Prove that the equation of the parallel through (0, 1) to the bisector of the angle XOY is $y=x+1$.

12. Through the point (0, 2) is drawn a straight line parallel to the line joining (0, 0) and (3, 4). Prove that the equation of the parallel is $4x-3y+6=0$.

Let P (x, y) be any point on the parallel. Let MP, the ordinate of P, cut the other line in Q.

Then
$$y=MP=MQ+QP$$
$$=\tfrac{4}{3}\,.\,OM+2.$$
$$\therefore\ 3y=4x+6.$$

13. The straight line through $(0, -1)$ parallel to the bisector of the angle YOX' is represented by the equation $x+y+1=0$.

14. Prove that the equation $4x-3y-4=0$ represents the parallel through (1, 0) to the line joining the origin and (3, 4).

15. The equation to the parallel through $(-3, 0)$ to the line joining (3, 4) to the origin is $4x-3y+12=0$.

16. Prove that the straight line $2x-3y=7$ passes through the point $(2, -1)$.

17. Which of the points (2, 1), $(-2, -2{\cdot}5)$, $(-5, -4)$, $(-1, 2)$ lie on the straight line $x-2y=3$?

18. The perpendicular through the origin to the line joining the origin to (3, 4) is represented by the equation $3x+4y=0$.

If A is the point (3, 4) and P any point (x, y) on the perpendicular through O to OA, and if BA and MP are the ordinates of A, P, then Δs OBA, PMO are similar.

19. Find the equation of the straight line joining (2, 3) and (3, 5).

20. Prove that $3x+4y=7$ is the equation of the straight line joining (1, 1) and $(9, -5)$.

21. Which of the following points do and which do not lie on the line $2x-3y=5$?—(1, 2), $(1, -1)$, $(-2, -3)$, $(-2, -4)$, $(3, \tfrac{1}{2})$, $(-3, -3)$, (7, 3).

22. Draw the straight lines whose equations are

(i) $x+y=2$, (ii) $2x-3y=5$, (iii) $3x+4y=7$,

(iv) $x-2y=1$, (v) $2x-y+3=0$.

CHAPTER III.

GRADIENT OF A STRAIGHT LINE. OBLIQUE AXES. POLAR COORDINATES. AREAS.

17. Gradient of a Straight Line. It has been agreed already (§ 9) that the positive direction of a straight line not parallel to either axis of coordinates is the direction in which a variable point (x, y) on the line travels when x increases. By the angle which a straight line makes with

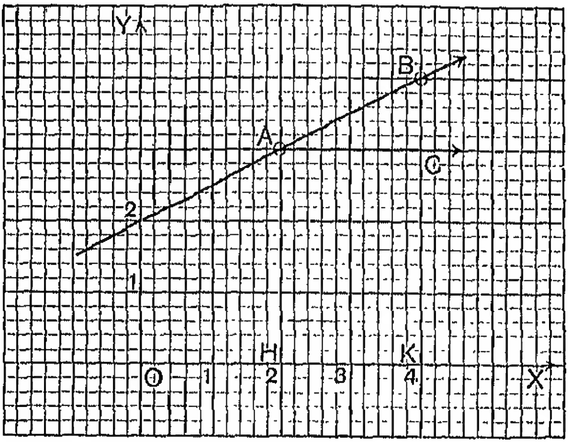


Fig. 15.

the x-axis, let us mean the acute angle θ measured *from* the positive direction of the x-axis *to* the positive direction of the line. If the line slopes up from left to right, θ is a *positive* acute angle; if the line slopes down from left to right θ is a *negative* acute angle. By the gradient of the

line we mean $\tan\theta$. Since the tangent of a positive acute angle is positive, and the tangent of a negative acute angle is negative, the gradient of a line may be positive or negative; it is positive when the line slopes up from left to right and negative when the line slopes down from left to right.

Ex. 1. Find the gradient of the line joining the points (2, 3) and (4, 4).

In Fig. 15, let A, B be the points (2, 3) and (4, 4).

Let the parallel to $X'OX$ through A meet the ordinate through B in C. Then θ=angle measured from AC to AB;

$$\therefore \tan\theta=\frac{CB}{AC}=\frac{4-3}{4-2}=\frac{1}{2}.$$

$\therefore$ the gradient of the line is 1/2.

We may say the line *rises* 1 in 2.

Ex. 2. Find the gradient of the line joining the points $(-5, 2)$ and $(7, -4)$.

In Fig. 16, let A, B be the points $(-5, 2)$ and $(7, -4)$.

Let the parallel to $X'OX$ through A meet the ordinate through B in C.

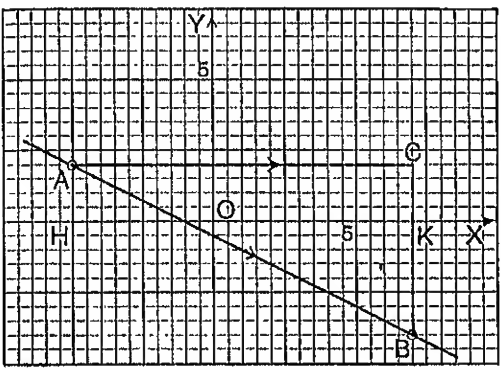


FIG. 16.

Then θ=angle measured from AC to AB.

$$\therefore \tan\theta=\frac{CB}{AC} \text{ (}CB\text{ is negative, }AC\text{ is positive)}$$

$$=\frac{-6}{12}=-\frac{1}{2}.$$

$\therefore$ the gradient of the line is $-1/2$.

We may say the line *falls* 1 in 2.

Ex. 3. Prove that the gradient of the line joining

(i) (3, 4) and (5, 7) is 3/2, (ii) (−3, 4) and (5, −7) is −11/8,

(iii) (0, 0) and (1, 2) is 2, (iv) (0, 0) and (−4, 3) is −3/4.

18. Formula for Gradient. *The gradient of the straight line passing through the points* (x_1, y_1) *and* (x_2, y_2) *is* $(y_2-y_1)/(x_2-x_1)$.

First. Of x_1 and x_2, let x_2 be the greater.

Let A be the point (x_1, y_1) (Figs. 15, 16).

" B " " (x_2, y_2).

Let the parallel to $X'OX$ through A meet the ordinate through B in C.

Then, since x_2 is greater than x_1,

the direction AB is the positive direction of the line,

the direction AC is the positive direction of the x-axis;

therefore θ, the angle which the line makes with the x-axis, is the angle measured from AC to AB.

$$\therefore \quad \tan\theta = \frac{CB}{AC}, \text{ in sign and magnitude.}$$

Now let HA, KB be the ordinates of A, B.

$$CB = KB - KC = KB - HA = y_2 - y_1;$$

$$AC = HK = OK - OH = x_2 - x_1.$$

$$\therefore \quad \tan\theta = \frac{CB}{AC} = \frac{y_2-y_1}{x_2-x_1}.$$

But the gradient of the line is $\tan\theta$ (§ 17).

$$\therefore \quad \text{the gradient of the line} = \frac{y_2-y_1}{x_2-x_1}.$$

Second. Of x_1 and x_2, let x_1 be the greater; then, by above,

$$\text{the gradient of the line} = \frac{y_1-y_2}{x_1-x_2} = \frac{y_2-y_1}{x_2-x_1}.$$

Hence the formula *always* holds.

For example, let (x_1, y_1) be $(-5, 2)$, and let (x_2, y_2) be $(7, -4)$. Then the gradient of the line joining these points is $\dfrac{y_2-y_1}{x_2-x_1} = \dfrac{-4-2}{7-(-5)} = -\dfrac{1}{2}$.

Ex. Find, by the formula, the gradients of the lines in Ex. 3, § 17.

19. Parallel Lines and Perpendicular Lines.

Let m_1, m_2 be the gradients of two straight lines. If the lines are parallel $m_1 = m_2$, if the lines are perpendicular $m_1 m_2 = -1$; and conversely.

Let the lines make with the x-axis angles θ_1, θ_2 (§ 17). If the lines are parallel, $\theta_1 = \theta_2$;

therefore $$\tan \theta_1 = \tan \theta_2,$$

that is, $$m_1 = m_2.$$

If the lines are perpendicular, one gradient is positive, the other is negative. Hence the product of the gradients is negative. It remains to show that the product is *numerically* equal to unity.

Let the lines cut the x-axis in A, B and one another in C (Fig. 17).

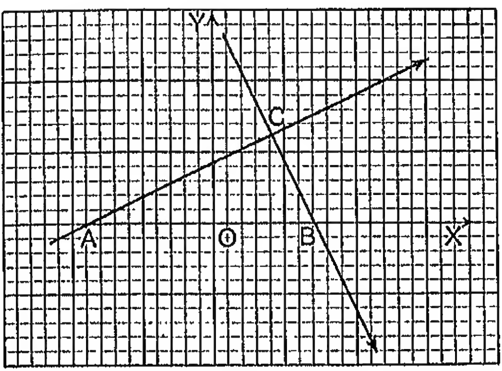


FIG. 17.

Then, $$\text{numerically, one gradient} = \frac{BC}{AC},$$

$$\text{numerically, other gradient} = \frac{AC}{BC}.$$

Therefore, numerically, product of gradients $= 1$.

But we have seen that the product is negative in sign.

$$\therefore\ m_1 m_2 = -1.$$

EXERCISES V.

1. Draw a line through the origin which rises 3 in 4.

2. Draw a line through the origin which falls 7 in 8.

3. Prove that the join of (1, 2) and (4, 7) rises 5 in 3.

4. Prove that the join of $(-2, -5)$ and $(4, -3)$ rises 1 in 3.

5. Prove that the join of $(-3, -4)$ and $(2, -5)$ falls 1 in 5.

6. What are the gradients of the lines in Exs. 1–5?

7. Prove that the line joining (x_1, y_1) and (x_2, y_2) rises (y_2-y_1) in (x_2-x_1), in algebraic measure.

8. Prove that the line joining (0, 0) and (3, 1) is parallel to the line joining $(-3, -6)$ and $(3, -4)$.

9. Prove that the join of (0, 0) and $(5, -1)$ has the same gradient as the join of $(-2, -3)$ and $(3, -4)$.

10. Prove that the three points (0, 3), (2, 7), (3, 9) are collinear.

11. Prove that the three points $(-4, 6)$, $(1, 1)$, $(6, -4)$ are collinear.

12. Prove that the join of (0, 0) and (4, 7) and the join of (0, 0) and $(7, -4)$ are perpendicular.

13. Prove that the join of (3, 2) and (7, 9) and the join of (3, 2) and $(10, -2)$ are perpendicular.

14. Prove that the join of $(-4, 7)$ and $(2, 5)$ and the join of $(-3, -5)$ and $(-2, -2)$ are perpendicular.

15. Prove that (2, 1), (6, 8), $(9, -3)$ are three vertices of a rectangle, and find the fourth.

*16. Prove that the straight line $y=\frac{3}{4}x$ rises 3 in 4.

17. Prove that the straight line $y=-\frac{7}{8}x$ falls 7 in 8.

18. Prove that the straight line $y=mx$ passes through the origin and has a gradient m.

19. Draw the straight line whose equation is

(1) $y=2x$; (2) $y=\frac{3}{5}x$; (3) $y=-\frac{3}{4}x$; (4) $3x-4y=0$;
(5) $4x+5y=0$.

20. Find what straight line is represented by the equation $\dfrac{y-3}{x-2}=\dfrac{1}{2}$.

By the formula, gradient$=\dfrac{y_2-y_1}{x_2-x_1}$, we see that the gradient of the line joining (2, 3) and (x, y) is $\frac{1}{2}$, $\therefore$ the join of (2, 3) to (x, y) rises 1 in 2, $\therefore$ if we draw through (2, 3) the straight line which rises

*The student who finds Exs. 16, 17, etc., difficult should read §§ 23, 24.

1 in 2, (x, y) must lie on this line. Hence this is the line represented by $(y-3)/(x-2)=\frac{1}{2}$.

21. Find what straight line is represented by the equation $\frac{y-2}{x-3}=\frac{1}{3}$. (The straight line through (3, 2) of gradient $\frac{1}{3}$.)

22. Find what straight line is represented by the equation $y-1=\frac{1}{2}(x-3)$.

23. Prove that the equation $\frac{y-2}{x}=3$ represents the straight line which is drawn through the point (0, 2) and has gradient 3.

24. What straight lines are represented by the following equations:

(1) $\frac{y-3}{x}=\frac{1}{2}$; (2) $y-2=\frac{1}{3}x$; (3) $y=\frac{1}{2}x+1$; (4) $y=-\frac{3}{4}x+2$;

(5) $y=-2x-3$?

25. Prove that the points (1, 2), (−5, −2) lie on the straight line represented by the equation $2x-3y+4=0$.

26. Which of the following points lie and which do not lie on the line whose equation is $3x-y=7$; viz.:

(0, 4), (−3, −10), (2, −1), (−1, −2), (−7, 4), (3, 2)?

27. Prove that the straight line $\frac{y-2}{x-1}=3$ passes through (1, 2) and has a gradient 3.

28. Find the equation of the straight line through (2, 3) of gradient $1\frac{1}{2}$.

29. Find the equation of the straight line through $(0, c)$ of gradient m.

30. Find the equation of the straight lines through

(i) (2, 5) of gradient $-\frac{2}{3}$; (ii) (−3, −2) of gradient $\frac{1}{2}$;

(iii) (−4, 2) of gradient $\frac{3}{5}$; (iv) (5, −3) of gradient −2.

31. Prove that the equation of the straight line through (x_1, y_1) of gradient m is $(y-y_1)/(x-x_1)=m$.

32. The vertex A of a triangle is the point (2, 5) and the gradient of the base BC is $\frac{1}{2}$, find the equation of the perpendicular from A to BC.

33. A, B, C, the vertices of a triangle, are the points (−5, 2), (1, 7), (3, −2) respectively. Find the gradients of BC, CA, AB and the equations of the perpendiculars from the vertices to the opposite sides.

34. A, B, C, the vertices of a triangle, are the points (7, 2), (−5, −2), (1, −9). Through A, B, C are drawn parallels to BC, CA, AB respectively; find the equations of the parallels.

35. Prove that (−1, 1), (5, 3), (11, 9), (5, 7) are the vertices of a parallelogram, and find the lengths of its diagonals.

36. Prove that $(-1, 1)$, $(2, 5)$, $(-5, 4)$ are the vertices of a right-angled triangle, and find the length of the hypotenuse.

37. Prove that $(2, -3)$, $(6, 1)$, $(2, 5)$, $(-2, 1)$ are the vertices of a square.

38. If P, Q are the points (a, b), $(-b, a)$, prove that OP is equal and perpendicular to OQ, where O is the origin.

39. If P, Q are the points (a, b), $(b, -a)$, prove that OP is equal and perpendicular to OQ, where O is the origin.

40. If P is the point $(3, 2)$ and if PQ is drawn perpendicular and equal to OP (where O is the origin), find the coordinates of Q.

41. If P is the point (a, b) and PQ is drawn perpendicular to and equal to OP (where O is the origin), find the coordinates of Q.

42. If P, Q are the points (a, b), (c, d) respectively and PR is drawn perpendicular and equal to PQ, find the coordinates of R.

20. Oblique Axes. It is sometimes convenient to take as axes of reference two lines $X'OX$, $Y'OY$ that are not at right angles; the axes are then said to be *oblique*.* The angle, ω say, between the axes is the angle through which $X'OX$ must be turned in order to be brought into coincidence with $Y'OY$. (Fig. 18.)

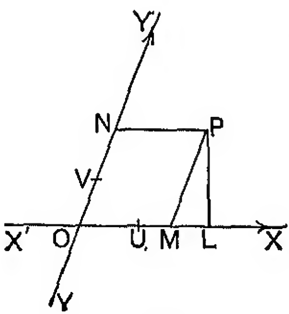


FIG. 18.

The only change on the construction of § 8 is that PM is drawn parallel to the y-axis (not perpendicular to the x-axis) and PN is drawn parallel to the x-axis (not perpendicular to the y-axis). If $OM = NP = x$ and $ON = MP = y$, then x is the abscissa and y the ordinate of P. The nomenclature is the same as that of § 8.

If L is the projection of P on $X'OX$ and $OL = x'$, $LP = y'$, then x', y' are the coordinates of P with reference to $X'OX$, and the axis through O *perpendicular* to $X'OX$; obviously

$$x' = x + y\cos\omega, \quad y' = y\sin\omega. \quad \text{..............(1)}$$

$$x = x' - y'\cot\omega, \quad y = y'\operatorname{cosec}\omega \text{..............(2)}$$

*This article may be postponed till Chapter IV. has been read.

The Section-Formulae of §§ 10, 14 are easily seen to be true for oblique axes, and the proof given in § 15 that the equation of the first degree represents a straight line is also applicable when the axes are oblique. When the equation of the straight line is written in the form

$$y = mx + c$$

the coefficient m is not equal to $\tan\theta$, where θ is the angle which the line makes with the x-axis (§ 17). If in Figs. 15, 16 we suppose the axes to be inclined at the angle ω, then m is, exactly as in § 18, equal in sign and magnitude to $\dfrac{CB}{AC}$. Now, by the sine-rule for triangles we have, so far as magnitude is concerned,

$$\frac{CB}{AC} = \frac{\sin CAB}{\sin ABC}.$$

When θ is positive, $\angle ABC = \omega - \theta$; when θ is negative the numerical value of $\angle CAB$ is $-\theta$, and $\angle ABC$ is the supplement of $\omega - \theta$. In both cases we have, in sign and magnitude,

$$m = \frac{CB}{AC} = \frac{\sin\theta}{\sin(\omega - \theta)}.$$

The equation of the line through (x_1, y_1) making the angle θ with the x-axis is,

$$y - y_1 = \frac{\sin\theta}{\sin(\omega - \theta)}(x - x_1). \qquad\qquad (3)$$

If $\omega = 90°$ the axes are rectangular, and we get $\tan\theta$ as the coefficient of $(x - x_1)$.

The equation $y' = x'\tan\theta$ in rectangular coordinates becomes, by equation (1),

$$y\sin\omega = (x + y\cos\omega)\tan\theta,$$

that is, $\quad y(\sin\omega\cos\theta - \cos\omega\sin\theta) = x\sin\theta,$

or $$y = \frac{\sin\theta}{\sin(\omega - \theta)}x.$$

We have thus another proof of the value of m in terms of θ and ω. It may be noted that

$$m=\frac{\sin\theta}{\sin(\omega-\theta)} \text{ gives } \tan\theta=\frac{m\sin\omega}{1+m\cos\omega}.$$

The expression for OP^2 (Fig. 18) is

$$OP^2=OM^2+MP^2-2OM\,.\,MP\cos OMP$$
$$=x^2+y^2+2xy\cos\omega,\ \dots\dots\dots\dots(4)$$

because in that figure $\angle OMP=180°-\omega$. The student will find that this formula holds for all positions of P; if ω is acute, the angle OMP of the triangle OMP is obtuse or acute according as OM and MP have the same sign or opposite signs, the position being reversed when ω is obtuse.

The general distance formula becomes

$$PQ^2=(x_1-x_2)^2+(y_1-y_2)^2+2(x_1-x_2)(y_1-y_2)\cos\omega\dots(5)$$

This formula may be readily obtained from that of §9. If the rectangular coordinates of P and Q are (x_1', y_1') and (x_2', y_2') then

$$PQ^2=(x_1'-x_2')^2+(y_1'-y_2')^2;$$

but, by equation (1), $x_1'=x_1+y_1\cos\omega$, $y_1'=y_1\sin\omega$, etc., so that we get

$$PQ^2=\{(x_1-x_2)+(y_1-y_2)\cos\omega\}^2+(y_1-y_2)^2\sin^2\omega,$$

which leads at once to equation (5).

Ex. If $y=m_1x+c_1$, $y=m_2x+c_2$ are the equations of two straight lines referred to axes inclined at the angle ω, then

(i) the lines are parallel if $m_1=m_2$;

(ii) the lines are perpendicular if $1+m_1m_2+(m_1+m_2)\cos\omega=0$.

Let the lines make angles α, β respectively with the x-axis, the meaning of angle being that given in §17. When $\alpha=\beta$, then obviously $m_1=m_2$. When the lines are perpendicular one of the angles α, β is positive and the other negative; suppose β to be positive, then $\beta-\alpha=90°$.

Now $$m_1=\frac{\sin\alpha}{\sin(\omega-\alpha)},\quad m_2=\frac{\sin\beta}{\sin(\omega-\beta)}=\frac{\cos\alpha}{-\cos(\omega-\alpha)},$$

and therefore, solving each equation for $\tan\alpha$, we get

$$\tan\alpha=\frac{m_1\sin\omega}{1+m_1\cos\omega}=-\frac{1+m_2\cos\omega}{m_2\sin\omega};$$

whence $$(1+m_1\cos\omega)(1+m_2\cos\omega)+m_1m_2\sin^2\omega=0,$$

or $$1+(m_1+m_2)\cos\omega+m_1m_2=0.$$

21. Polar Coordinates. The position of P (Fig. 6) will be known when we are given (i) the distance r of P from the origin O, and (ii) the angle θ which the step OP makes with the positive direction of the x-axis, that is, the angle through which OX (not OX') must be turned till it coincides with OP. These numbers r and θ are called the **polar coordinates** of P with reference to the **pole** or **origin** O and the **initial line** OX; r is the **radius vector** and θ the **vectorial angle** of P.

If the rectangular coordinates of P are x and y, then we have

$$x=r\cos\theta,\quad y=r\sin\theta. \quad\text{.................. (1)}$$

It is usual to suppose r to be always positive; $\cos\theta$ and $\sin\theta$ have then the same signs as x and y respectively. We may, however, allow r to take negative values, provided that when r or OP is negative we take θ to be the angle that the step PO (not OP) makes with OX.

From equation (1) we find

$$r=\sqrt{(x^2+y^2)};\quad \tan\theta=\frac{y}{x}. \quad\text{................ (2)}$$

In determining θ the equation $\tan\theta=y/x$ is not sufficient by itself; we must remember that (r being positive) the signs of $\cos\theta$ and $\sin\theta$ are the same as those of x and y respectively.

We shall make little use of polar coordinates in this book.

Ex. 1. What are the Cartesian coordinates of the points whose polar coordinates are:

(i) $(5, 30°)$; (ii) $(5, 120°)$; (iii) $(5, 270°)$?

Applying the formulae $x=r\cos\theta$, $y=r\sin\theta$, we find

(i) $x=\frac{5\sqrt{3}}{2}$, $y=\frac{5}{2}$; (ii) $x=-\frac{5}{2}$, $y=\frac{5\sqrt{3}}{2}$; (iii) $x=0$, $y=-5$.

Ex. 2. The general equation of a straight line when referred to polar coordinates is of the form

$$\frac{c}{r}=a\cos\theta+b\sin\theta.$$

In the Cartesian equation $ax+by=c$, put $r\cos\theta$ for x and $r\sin\theta$ for y; we then get the form stated. The student will readily see that the following are equivalent forms, α, β, p, q being constants:

$$r\cos(\theta+\alpha)=p, \quad r\sin(\theta+\beta)=q.$$

22. Areas. Let A, B (Fig. 19) be the points (x_1, y_1), (x_2, y_2) referred to rectangular axes $X'OX$, $Y'OY$, and let (r_1, θ_1), (r_2, θ_2) be their polar coordinates, O being the pole and OX the initial line; then

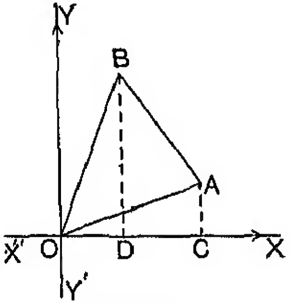


Fig. 19.

$$x_1=r_1\cos\theta_1, \quad y_1=r_1\sin\theta_1;$$
$$x_2=r_2\cos\theta_2, \quad y_2=r_2\sin\theta_2.$$

The angle AOB is equal to $(\theta_2-\theta_1)$, and the area of the triangle OAB is

$$\tfrac{1}{2}r_1r_2\sin(\theta_2-\theta_1).$$

But $\quad \sin(\theta_2-\theta_1)$
$$=\cos\theta_1\sin\theta_2-\cos\theta_2\sin\theta_1,$$

and therefore, denoting by $\triangle OAB$ the area of the triangle OAB, we have

$$\triangle OAB=\tfrac{1}{2}(r_1\cos\theta_1 . r_2\sin\theta_2-r_2\cos\theta_2 . r_1\sin\theta_1)$$
$$=\tfrac{1}{2}(x_1y_2-x_2y_1). \quad \text{.............................}(1)$$

This formula may also be proved in the following way. Let C and D be the projections of A and B on the x-axis; then the triangle OAB is equal to the sum of the triangle ODB and the quadrilateral $BDCA$ diminished by the triangle OCA. Therefore

$$\triangle OAB=\triangle ODB+\text{quad. } BDCA-\triangle OCA$$
$$=\tfrac{1}{2}OD . DB+\tfrac{1}{2}DC(DB+CA)-\tfrac{1}{2}OC . CA$$
$$=\tfrac{1}{2}x_2y_2+\tfrac{1}{2}(x_1-x_2)(y_2+y_1)-\tfrac{1}{2}x_1y_1$$
$$=\tfrac{1}{2}(x_1y_2-x_2y_1).$$

We have thus obtained an expression for the area of the triangle OAB in terms of the coordinates of A and B; let us apply the formula to two simple cases.

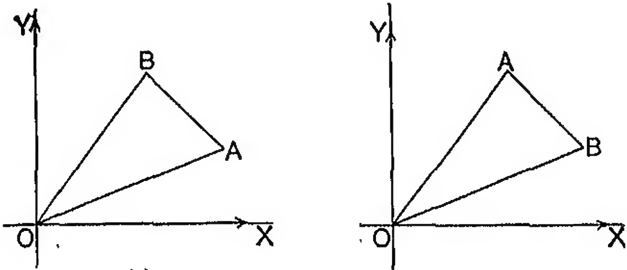
(i) Let A be the point (5, 2) and B the point (3, 4); we find

$$\triangle OAB = \tfrac{1}{2}(5 \times 4 - 3 \times 2) = 7.$$

(ii) Let A be the point (3, 4) and B the point (5, 2). This triangle is the same as in case (i), but the letters attached to the points (5, 2) and (3, 4) have been interchanged; we find

$$\triangle OAB = \tfrac{1}{2}(3 \times 2 - 5 \times 4) = -7.$$

The numerical value of the area is thus the same as in case (i), but the number that measures the area is now *negative.* If it be remembered that coordinates are the measures of steps, and therefore involve direction as well as magnitude, it is not a matter for surprise that a calculation which involves coordinates should result in a negative number; we may conjecture that the above difference in sign will have some connection with the two different senses in which the lines that bound the triangle may be traced.



(*a*) (*b*)

Fig. 20.

Fig. 20 (*a*) shows the triangle of case (i) and Fig. 20 (*b*) the triangle of case (ii). If we describe the boundary of the triangle OAB in the order in which the letters O, A, B are written, it will be seen that in case (i) (Fig. 20 (*a*)) the area lies on our left hand, while in case (ii) (Fig. 20 (*b*)) the area lies on our right hand.

We shall now state a rule which is not hard to prove by examining the sign of

$$\tfrac{1}{2}r_1r_2\sin(\theta_2-\theta_1),$$

but which it will be sufficient at this stage for the student to verify by testing it for positions of A and B in each of the four quadrants.

Rule. If A is the point (x_1, y_1) and B the point (x_2, y_2), O being the origin of coordinates, the numerical value of the expression $\frac{1}{2}(x_1y_2-x_2y_1)$ always gives the magnitude of the area of the triangle OAB; the sign of the expression will be positive if, when we describe the boundary in the order O, A, B, the area lies on our left hand, but the sign will be negative when the area lies on our right hand.

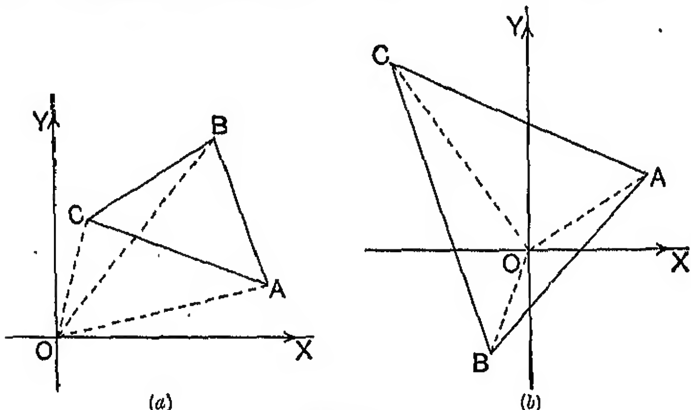


FIG. 21.

We shall now use the symbol $\triangle OAB$ to indicate the *algebraic* measure of the area of the triangle OAB, so that we have

$$\triangle OAB = \tfrac{1}{2}(x_1y_2-x_2y_1),\quad \triangle OBA = \tfrac{1}{2}(x_2y_1-x_1y_2).$$

In other words, an area is, like a step, a magnitude which may be either positive or negative.*

* In applying the formulae it is convenient to denote the coordinates of point P by the symbols x_P, y_P; thus;

$$\triangle OAB = \tfrac{1}{2}(x_Ay_B - x_By_A),\quad \triangle OBA = \tfrac{1}{2}(x_By_A - x_Ay_B).$$

Let us now consider the triangle ABC, the coordinates of whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) respectively (Figs. 21 (a) and (b)). We have the following equations:

$$\triangle OAB = \tfrac{1}{2}(x_1y_2 - x_2y_1),$$
$$\triangle OBC = \tfrac{1}{2}(x_2y_3 - x_3y_2),$$
$$\triangle OCA = \tfrac{1}{2}(x_3y_1 - x_1y_3).$$

In Fig. 21 (a) $\triangle OAB$ and $\triangle OBC$ are positive and $\triangle OCA$ negative; in Fig. 21 (b) all are negative. In both cases we have the relation

$$\triangle OAB + \triangle OBC + \triangle OCA = \triangle ABC, \quad \text{............(2)}$$

a relation which may be verified to hold whatever be the positions of the four points O, A, B, C. Inserting in equation (2) the values in terms of the coordinates, we find for the area of the triangle ABC the formula

$$\triangle ABC = \tfrac{1}{2}\{x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3\}$$
$$= \tfrac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}. \quad \text{...(3)}$$

The second of these forms is perhaps the more easily remembered.

The formula (3) gives the area in sign and magnitude.

The student may prove that if the axes are inclined at the angle ω the area is equal to

$$\tfrac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}\sin\omega. \quad \text{...............(4)}$$

He may do this by showing that equation (1) becomes

$$\triangle OAB = \tfrac{1}{2}(x_1y_2 - x_2y_1)\sin\omega.$$

Ex. 1. Find the area of the triangle the coordinates of whose vertices, taken in order, are $(2, 7)$, $(5, -1)$, $(-1, -4)$.

The area is, by formula (3),

$$\tfrac{1}{2}\{2(-1+4) + 5(-4-7) + (-1)(7+1)\} = -28{\cdot}5.$$

Ex. 2. Find the area of the quadrilateral $ABCD$, the coordinates of A, B, C, D being $(2, 1)$, $(-2, 2)$, $(-1, -1)$, $(5, -2)$.

The quadrilateral is the sum of the triangles ABC and ACD.

$$\triangle ABC = \tfrac{1}{2}\{2(2+1) + (-2)(-1-1) + (-1)(1-2)\} = 5\tfrac{1}{2},$$
$$\triangle ACD = \tfrac{1}{2}\{2(-1+2) + (-1)(-2-1) + 5(1+1)\} = 7\tfrac{1}{2}.$$

The area of the quadrilateral is therefore 13.

Ex. 3. If "quad. $ABCD$" denotes the area of the quadrilateral $ABCD$ in sign and magnitude, show that

$$\text{quad. } ABCD = \triangle OAB + \triangle OBC + \triangle OCD + \triangle ODA.$$

The proof is a very obvious extension of that given in the text for the relation numbered (2); it can clearly be extended to any polygon.

Ex. 4. Plot the points $A(2, 0)$, $B(8, 0)$, $C(8, -2)$, $D(2, 5)$ and join AB, BC, CD, DA.

It will be noticed that CD crosses AB between A and B; a quadrilateral of this kind is called a **cross-quadrilateral**. Its "area" calculated by the rule of Example 3 is

$$\triangle OAB + \triangle OBC + \triangle OCD + \triangle ODA = 0 + (-8) + (22) + (-5) = 0.$$

Ex. 5. Plot the points $A(a, 0)$, $B(b, 0)$, $C(b, c)$, $D(a, d)$, and show that the area of the quadrilateral $ABCD$, whether "cross" or not, is $\frac{1}{2}(b-a)(d+c)$ or $\frac{1}{2}AB(AD+BC)$, where AB, AD, BC are steps.

EXERCISES VI.

Calculate the area of each of the triangles and polygons whose vertices are specified in Examples 1-9; the perimeter is to be traced in the order in which the vertices are named.

1. $(5, 7)$, $(-3, 4)$, $(0, -6)$. **2.** $(3, 1)$, $(4, -2)$, $(-1\ -2)$.

3. $(1, 5)$, $(6, -3)$, $(-3, -4)$. **4.** (x, y), $(0, b)$, $(a, 0)$.

5. $(4, 4)$, $(-3, 5)$, $(-5, -5)$, $(5, -3)$.

6. $(2, 3)$, $(5, -2)$, $(-2, -4)$, $(-5, 0)$.

7. $(3, 1)$, $(1, 4)$, $(-3, 2)$, $(-2, -2)$, $(2, -3)$.

8. $(2, -1)$, $(6, -1)$, $(-1, 1)$, $(3, 3)$.

9. $(4, 1)$, $(-2, 5)$, $(0, -2)$, $(2, 5)$, $(-4, 1)$.

10. The coordinates of A, B, C are $(6, 3)$, $(-3, 5)$, $(4, -2)$ respectively and P is the point (x, y); show that

$$\frac{\triangle PBC}{\triangle ABC} = \frac{x+y-2}{7}.$$

11. B and C are any two points on the straight line given by the equation $ax+by+c=0$, and $P(x_1, y_1)$, $Q(x_2, y_2)$ are any two points that do not lie on the line; show by considering the sign of the areas of the triangles PBC, QBC that the expressions

$$ax_1+by_1+c \text{ and } ax_2+by_2+c$$

are of the same sign or of opposite signs according as P and Q are on the same side or on opposite sides of the line.

12. If (x, y) is *any* point collinear with (x_1, y_1) and (x_2, y_2), prove that

$$x(y_1 - y_2) - y(x_1 - x_2) + x_1 y_2 - x_2 y_1 = 0,$$

and find the equation of the join of (2, 5) and (−7, 1).

13. From the formula for the area of a triangle deduce that if a variable point (x, y) moves on a straight line, then $Ax + By + C = 0$, where A, B, C are constants.

CHAPTER IV.

REPRESENTATION OF GEOMETRICAL LOCI BY ANALYTICAL EQUATIONS. THE STRAIGHT LINE.

23. The Equation y=mx. Axes and scale-units being chosen or assigned, a straight line through the origin is completely specified when its gradient (§ 17) is specified. But we know that the same straight line may be represented by an equation in x, y. Let a straight line through the origin have gradient m; it is required to translate the defining conditions of the line into an equation.

Let $Q(x, y)$ be any point on the line (Fig. 14). Let M be the projection of Q on $X'OX$.

The gradient of the line is $\frac{MQ}{OM}$ or $\frac{y}{x}$.

$$\therefore \frac{y}{x}=m;$$

$$\therefore y=mx.$$

Since (x, y) is *any* point on the line,

$$\mathbf{y=mx}$$

is the equation of the line.

24. The Equation y=mx+c. Let a straight line be specified by its gradient m and its intercept c on the y-axis; to find its equation.

Let $P(x, y)$ be any point on the line (Fig. 14).
Let the line cut the y-axis in C.
Let OQ be the parallel to the line through the origin.
Let MP, the ordinate of P, cut the parallel in Q.

Since the lines are parallel, their gradients are equal (§ 19).

$$\therefore \text{ gradient of } OQ=m;$$

$$\therefore \frac{MQ}{OM}=m;$$

$$\therefore MQ=m\,.\,OM.$$

But $$MP=MQ+QP;$$

$$\therefore MP=m\,.\,OM+OC;$$

$$\therefore \quad y=mx+c.$$

But (x, y) is any point on the line; hence

$$\mathbf{y=mx+c}$$

is the equation of the specified line.

EXERCISES VII.

1. Find the equation of the straight line which passes through the point (0, 2) and rises 5 in 3.

$$m=\text{gradient}=\tfrac{5}{3}; \quad c=2;$$

hence $y=mx+c$ becomes $y=\frac{5}{3}x+2$ or $5x-3y+6=0$, which is the required equation (see Fig. 14).

2. Find the equation of the straight line which passes through the point $(0, -2)$ and falls 4 in 5.

$$m=\text{gradient}=-\tfrac{4}{5}; \quad c=-2;$$

$\therefore$ $y=mx+c=-\frac{4}{5}x-2$; that is, $4x+5y+10=0$ is the equation of the line.

3. Find the equation of the straight line of gradient $-\frac{3}{4}$ whose intercept on the y-axis is $1\frac{1}{3}$.

4. Find the equation of the straight line of gradient $\frac{3}{5}$ whose intercept on the y-axis is $-2\frac{1}{2}$.

5. Find the equation of the straight line drawn through (0, 2) to make an angle of 30° with $X'OX$.

6. Find the equation of the straight line whose intercept on the y-axis is $-2{\cdot}2$ and which makes an angle of $-60°$ with $X'OX$.

7. Draw the graph of

(i) $y=2x+1$; (ii) $y=-3x+2$; (iii) $y=\frac{3}{5}x-5$;
(iv) $y=-\frac{4}{3}x-1$; (v) $2y=3x+4$.

[(i) passes through (0, 1) and rises 2 in 1.]

8. What are the gradients of the following lines?

(i) $y=2x+3$; (ii) $y=-2x-1$; (iii) $y=-\frac{3}{4}x+7$;
(iv) $2y=x+2$; (v) $3y=-2x+4$; (vi) $2y-x=3$;
(vii) $7y+3x+1=0$; (viii) $3x-4y+5=0$; (ix) $7x+5y=0$.

9. What is the gradient of the line $ax+by+c=0$?

10. What is the gradient of the line $x/a+y/b=1$?

11. What is the gradient of the line $(y-3)=m(x-2)$, and of the line $y-y_1=m(x-x_1)$?

12. Use the gradient formula, viz. $\frac{y_2-y_1}{x_2-x_1}=m$, to establish the equation $y=mx+c$.

$\frac{y-c}{x-0}$=gradient of the line joining $(0, c)$ and (x, y); hence $\frac{y-c}{x}=m$ or $y=mx+c$.

13. Draw the graphs of

(i) $2x-y+1=0$; (ii) $3x+4y=7$; (iii) $5x=2y+3$;
(iv) $3x+2y+4=0$.

25. The Linear Equation. Every straight line, considered with reference to a system of rectangular axes and scale-units, has a definite gradient and makes a definite intercept on the y-axis. Hence every straight line may be represented by an equation of the form $y=mx+c$, which is an equation linear in x, y (§ 15). Conversely, any equation of the form $y=mx+c$ represents a straight line. To prove this we have only to reverse the steps of § 24. Thus, let P (Fig. 14) be *any* point on the graph or locus of $y=mx+c$. Through the origin draw the straight line OQ of gradient m. Let C be the point $(0, c)$ and let MP, the ordinate of P, meet OQ in Q. Then

$$m=\text{gradient of } OQ$$
$$=\frac{MQ}{OM};$$
$$\therefore\ MQ=m\,.\,OM.$$

But
$$y=mx+c;$$
$$\therefore\ MP=m\,.\,OM+OC;$$
$$\therefore\ MP=MQ+OC;$$

$$\therefore\ MQ+QP=MQ+OC;$$

$$\therefore\ QP=OC.$$

Also QP is parallel to OC.

$$\therefore\ CP \text{ is parallel to } OQ.$$

Therefore the locus of P is the straight line through the fixed point C parallel to the fixed line OQ.

Now the general linear equation $Ax+By+C=0$ may be written in the form

$$y=\left(-\frac{A}{B}\right)x+\left(-\frac{C}{B}\right),\ (B \neq 0),$$

i.e. in the form $y=mx+c$.

Hence the conclusion: *any straight line may be represented by a linear equation in x, y, and, conversely, any linear equation in x, y represents a straight line.*

The cases of lines parallel to the axes may be treated separately; a similar conclusion holds.

26. The equation $\mathbf{y-y_1=m(x-x_1)}$. If we write the equation

$$y-y_1=m(x-x_1)$$

in the form

$$\frac{y-y_1}{x-x_1}=m,$$

we see (§ 18) that the gradient of the straight line joining (x_1, y_1) to (x, y), any point on the locus or graph represented by the equation, is the constant m.

Hence $$(\mathbf{y-y_1})=\mathbf{m(x-x_1)}$$

represents the straight line, of gradient m, through the fixed point (x_1, y_1).

27. The equation $\mathbf{y-y_1=\dfrac{y_1-y_2}{x_1-x_2}(x-x_1)}$. If we write the equation

$$y-y_1=\frac{y_1-y_2}{x_1-x_2}(x-x_1)$$

in the form

$$\frac{y-y_1}{x-x_1}=\frac{y_1-y_2}{x_1-x_2},$$

we see (§ 18) that the gradient of the straight line joining the point (x_1, y_1) to (x, y), any point on the locus or curve represented by the equation, is equal to the gradient of the line joining (x_1, y_1) and (x_2, y_2).

Hence $$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1)$$

represents the straight line through the points (x_1, y_1) and (x_2, y_2).

Ex. 1. Find the equation of the straight line which passes through the point (2, 3) and rises 2 in 1.

In the equation $$y - y_1 = m(x - x_1),$$
put $x_1 = 2$, $y_1 = 3$, $m = 2/1 = 2$.

The required equation is $y - 3 = 2(x - 2)$

or $$2x - y - 1 = 0 \text{ (Fig. 22).}$$

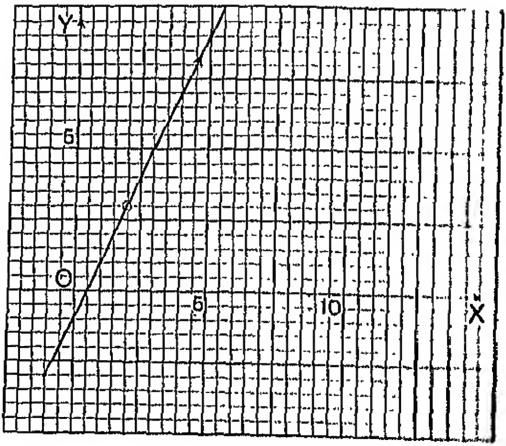


FIG. 22.

Ex. 2. Find the equation of the straight line which passes through the two points $(-4, 1)$ and $(7, -5)$.

In the equation $$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1),$$
put $x_1 = -4$, $y_1 = 1$, $x_2 = 7$, $y_2 = -5$.

We get $$y-1=\frac{1+5}{-4-7}(x+4),$$

$$\text{i.e.}\quad y-1=-\frac{6}{11}(x+4),$$

which becomes $$6x+11y+13=0 \text{ (Fig. 23)}.$$

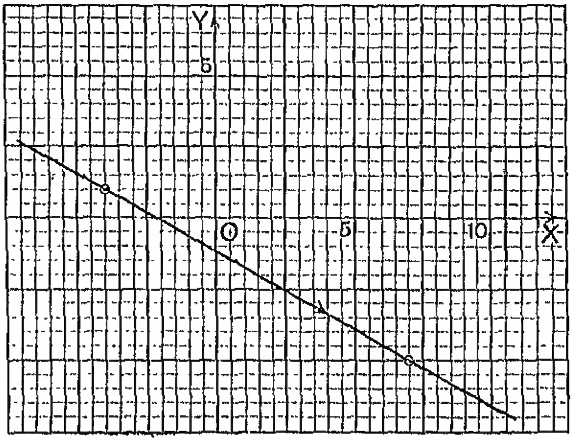


FIG. 23.

As a check, note that $6(-4)+11.1+13=0$,
and $6.7+11(-5)+13=0$.

EXERCISES VIII.

1. Find the equation of the straight line which passes through $(-5, -2)$ and falls 3 in 2.

2. Find the equations of the two straight lines which both pass through the point $(4, -5)$, one of gradient $3/4$, the other of gradient $-3/4$.

3. Find the equation of the join of $(2, 3)$ and $(7, 5)$.

4. Prove that the join of $(-3, 7)$ and $(-1, 5)$, and the join of $(2, -4)$ and $(-5, 3)$ are parallel, and find the equations of the lines.

5. Prove that the line joining the points $(-2, 6)$ and $(0, 4)$ is perpendicular to the line joining $(2, 4)$ and $(-5, -3)$, and find the equations of the lines.

6. Through (0, 1) is drawn the parallel to the line joining (0, 0), (3, 4); find the equation of the parallel.

[Use $y-y_1=m(x-x_1)$. $x_1=0$, $y_1=1$, $m=4/3$.]

7. What is the gradient of each of the following lines?

(i) $y=2x-3$; (ii) $y-3x=1$; (iii) $2y+4x=7$;
(iv) $2x-5y+1=0$; (v) $3x+2y-2=0$; (vi) $ax+by+c=0$.

8. Prove that the following pairs of lines are parallel:

(i) $2x-3y+1=0$ and $2x-3y-58=0$;
(ii) $x+2y=7$ and $2x+4y-13=0$;
(iii) $ax+by+c=0$ and $ax+by+d=0$.

9. Prove that the following pairs of lines are perpendicular:

(i) $2x+3y+1=0$ and $3x-2y+17=0$;
(ii) $5x-2y+3=0$ and $2x+5y-11=0$;
(iii) $ax+by+c=0$ and $bx-ay+d=0$.

10. Find the equations of the sides of the triangle whose vertices are (3, 2), (5, 7), (9, 1).

11. Find the equations of the medians of the triangle whose vertices are (−5, −2), (4, −6), (1, 7).

12. If A, B, C are the points (1, 5), (3, 1), (4, 8) respectively, what is the gradient of BC and of the perpendicular AD from A to BC? Find the equation of AD.

13. If A, B, C are the points (5, −3), (−5, 3), (4, 7) respectively, find the equation of the line joining the middle points of AB and AC.

14. Find the equation of the perpendicular bisector of the join of (2, 3) and (5, −2). Does the point (8, 4) lie on the bisector?

15. Which of the following sets of points are collinear?

(i) (1, 1), (2, 2), (6, 6); (ii) (5, 4), (0, 1), (4, −4);
(iii) (5, −2), (−4, 1), ($\frac{1}{2}$, 0); (iv) (1, 3), (−2, −6), (4, 12).

16. Find the point of intersection of the lines $2x-3y+1=0$, $x+y-2=0$.

(Solve the equations as simultaneous equations.)

17. Find the point of intersection of the join of (5, −2) and (−2, 4), and the join of (3, 7) and (−11, −2).

18. Find the orthocentre (*i.e.* intersection of perpendiculars) of triangle ABC of Ex. 10.

19. Find the orthocentre of triangle ABC of Ex. 11.

20. Find the intersection of the medians of triangles ABC in Exs. 10, 11.

28. Parallel through (h, k) to ax+by+c=0. In geometry it is frequently necessary to construct a parallel through a given point to a given straight line. If (h, k) is any given point and $ax+by+c=0$ is any given straight line, the following rule enables us to write down at once the equation of the parallel through (h, k) to $ax+by+c=0$.

Rule. *In the equation* $ax+by+c=0$: (i) *delete the absolute term* c; (ii) *replace* x *by* $(x-h)$ *and* y *by* $(y-k)$; *the equation*

$$a(x-h)+b(y-k)=0, \qquad \text{.....................(1)}$$

so obtained, is the parallel through (h, k) *to*

$$ax+by+c=0.$$

Proof. The gradient of the straight line $ax+by+c=0$ is $-a/b$.

The gradient of the straight line $a(x-h)+b(y-k)=0$ is $-a/b$.

Therefore the lines are parallel (§ 19).

Again the line (1) passes through (h, k) if

$$a(h-h)+b(k-k)=0,$$

and this is true. The rule is therefore proved.

Ex. 1. Find the equation of the parallel to $3x-2y+4=0$ through $(2, 7)$.

By the rule, the equation is

$$3(x-2)-2(y-7)=0,$$

that is
$$3x-2y+8=0.$$

Ex. 2. Find the equation of the parallel:

(i)	through	$(5, 3)$	to $2x-y+1=0$,
(ii)	"	$(3, 1)$	" $y=3x+4$,
(iii)	"	$(-1, 2)$	" $5x+4y-7=0$,
(iv)	"	$(2, -1)$	" $3x-2y-3=0$,
(v)	"	$(-3, -1)$	" $y=2x-1$,
(vi)	"	the origin	" $2x-7y+5=0$.

29. Perpendicular through (h, k) to ax+by+c=0. It is necessary to be able to write down the equation of the perpendicular let fall from a given point to a given straight line. The following rule is used for writing

down the equation of the perpendicular from (h, k) to $ax+by+c=0$:

Rule. *In the equation* $ax+by+c=0$: (i) *delete the absolute term* c; (ii) *change the sign before the term in* y; (iii) *interchange the coefficients of* x *and* y; (iv) *replace* x *by* $(x-h)$ *and* y *by* $(y-k)$; *the equation*

$$\mathbf{b(x-h)-a(y-k)=0}, \qquad \text{.....................(1)}$$

so obtained, is the perpendicular through (h, k) *to*

$$ax+by+c=0.$$

Proof. The gradient of $ax+by+c=0$ is $-a/b=m_1$, say. The gradient of $b(x-h)-a(y-k)=0$ is $b/a=m_2$, say.

And $$m_1m_2=-\frac{a}{b}\times\frac{b}{a}=-1;$$

therefore the lines are perpendicular (§ 19).

Again, the line (1) passes through (h, k) if

$$b(h-h)-a(k-k)=0,$$

and this is true. The rule is therefore correct.

EXERCISES IX.

1. Find the equation of the perpendicular through $(5, 1)$ to $2x-3y+4=0$.

Begin with $2x-3y+4=0$.

Following (i) of Rule, we get $2x-3y=0$,
" (ii) " " $2x+3y=0$,
" (iii) " " $3x+2y=0$,
" (iv) " " $3(x-5)+2(y-1)=0$, ...(A)
i.e. $3x+2y-17=0$.

With a little practice equation (A) can be written down at once.

2. Find the equation of the perpendicular:

(i) through $(5, 2)$ to $3x-4y+1=0$,
(ii) " $(3, 1)$ " $2x+5y+7=0$,
(iii) " $(2, 3)$ " $y=2x+1$,
(iv) " $(-2, 1)$ " $x+3y=4$,
(v) " $(2, -3)$ " $5x-3y=8$,
(vi) " $(-1, -2)$ " $7x+2y-2=0$,
(vii) " the origin " $3x-4y=5$.

3. Find the equations of the parallel and perpendicular through $(-4, 1)$ to $7x-2y+3=0$.

4. Find the equations of the parallel and perpendicular through $(-2, -5)$ to $3x+8y-4=0$.

5. Find the coordinates of the orthocentre of the triangle whose vertices are $(5, 2)$, $(-1, 1)$, $(2, 7)$.

6. Find the coordinates of the foot of the perpendicular let fall from $(1, 2)$ to $3x+4y+9=0$. Find also the length of the perpendicular.

30. Length of Perpendicular.

If a perpendicular be let fall from the point (x_1, y_1) to the straight line $ax+by+c=0$, then the numerical value of the length of the perpendicular is

$$\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}.$$

Let (x_2, y_2) in Fig. 24 be the foot of the perpendicular.

The equation of the perpendicular from (x_1, y_1) to $ax+by+c=0$ is (§ 29) $b(x-x_1)-a(y-y_1)=0$.

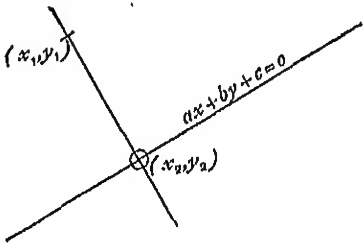


FIG. 24.

Now (x_2, y_2) lies on this line;

therefore $$b(x_2-x_1)-a(y_2-y_1)=0. \quad\text{.................}(1)$$

Again, (x_2, y_2) lies on the line $ax+by+c=0$;

therefore $$ax_2+by_2+c=0.$$

Subtract ax_1+by_1+c from both sides:

then $$a(x_2-x_1)+b(y_2-y_1)=-(ax_1+by_1+c). \quad ..(2)$$

But, by (1), $b(x_2-x_1)-a(y_2-y_1)=0.$

Square and add:

then $(a^2+b^2)\{(x_2-x_1)^2+(y_2-y_1)^2\}=(ax_1+by_1+c)^2$;

$$\therefore\ (x_2-x_1)^2+(y_2-y_1)^2=\frac{(ax_1+by_1+c)^2}{a^2+b^2}.$$

$\therefore$ by Distance-Formula (§ 9),

square of distance from (x_1, y_1) to $(x_2, y_2)=\dfrac{(ax_1+by_1+c)^2}{a^2+b^2}$;

i.e. length of perpendicular from (x_1, y_1) to $ax+by+c=0$ is the numerical value of

$$\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}.$$

EXERCISES X.

1. Find the perpendicular distance from the point $(-2, 3)$ to the straight line $3x-4y+2=0$.

Use the formula, perp. dist. $=\dfrac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$.

$$a=3,\ b=-4,\ c=2;\quad x_1=-2,\ y_1=3.$$

Therefore perp. dist. $=\dfrac{3(-2)+(-4).3+2}{\sqrt{3^2+(-4)^2}}$, numerically,

$$=\frac{16}{5}.$$

2. Find the distance *from* $(4, 7)$ *to* $3x-4y+2=0$.

3. Find the distance from $(4, 3)$ to the line $x=y$. *Verify geometrically.*

4. Find the distance from the origin to $x+y+1=0$. *Verify geometrically.*

5. *Prove that the points* $(1, 3)$ and $(-7, -3)$ *are equidistant from the line* $3x-4y+7=0$.

6. Find the *distance* between the *parallel* lines:

(i) $x+y+1=0$, $x+y-1=0$; (ii) $3x-4y+9=0$, $3x-4y-1=0$;

(iii) $2x-3y+4=0$, $4x-6y-7=0$.

7. The straight line $3x+4y-5=0$ touches a circle whose centre $(2, -3)$. Find the radius of the circle.

8. Prove that the lines $4x-3y+5=0$, $4x-3y-5=0$, $3x+4y+5=0$, $3x+4y-5=0$ all touch the circle whose centre is the origin and whose radius is unity.

9. Prove that one of the common tangents to the circle, centre (0, 0) and radius 2, and the circle centre (4, 0) and radius 1, is

$$3x+\sqrt{7}\,.\,y-8=0.$$

10. Show that $4x-3y=25$ touches the circles whose centres are (11, −2) and (−11, 2), and whose radii are 5 and 15 respectively.

11. Find the value of c if $3x+4y=c$ is a common tangent to the circles whose centres are (1, 3) and (−3, 1), and whose radii are 1 and 3 respectively.

12. Prove that the product of the perpendiculars from $(c, 0)$ and $(-c, 0)$ to the straight line $bx\cos\theta+ay\sin\theta=ab$ is b^2, where $a^2=b^2+c^2$.

13. Prove that the point (2, 2) is equidistant from the lines $2x-y+2=0$ and $x-2y+6=0$.

14. Prove that the point (1, 1) lies on one of the bisectors of the angles formed by the lines $4x-3y+1=0$ and $3x-4y+3=0$. Illustrate by a figure.

15. If the point (x_1, y_1) lies on a bisector of the angles formed by the lines $Ax+By+C=0$ and $ax+by+c=0$, prove that $\dfrac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}$ and $\dfrac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$ are numerically equal.

16. If the point (x_1, y_1) lies on one of the bisectors of the angles formed by the lines $7x-5y+1=0$ and $5x+7y+3=0$, prove that $7x_1-5y_1+1$ and $5x_1+7y_1+3$ are numerically equal.

17. If the point (x_1, y_1) lies on one of the bisectors of the angles formed by the lines $4x-3y+2=0$ and $3x+4y+3=0$, prove that $4x_1-3y_1+2=\pm(3x_1+4y_1+3)$.

18. If the point (x_1, y_1) lies on one of the bisectors of the angles formed by the lines $3x-y+2=0$ and $x+3y+3=0$, prove that either $2x_1-4y_1=1$ or else $4x_1+2y_1+5=0$.

19. Prove that the equations of the two bisectors of the angles formed by the lines $5x-3y+2=0$ and $3x-5y+5=0$ are

$$5x-3y+2=\pm(3x-5y+5).$$

20. Prove that the bisectors of the angles formed by the lines $2x+3y+1=0$ and $x-y+4=0$ are given by the equations

$$\frac{2x+3y+1}{\sqrt{13}}=\pm\frac{x-y+4}{\sqrt{2}}.$$

21. Prove that $x+y=3$ and $x-y+1=0$ are the bisectors of the angles formed by the lines $3x-2y+1=0$ and $2x-3y+4=0$.

22. Prove that the bisectors of the angles formed by the lines $ax+by+c=0$ and $Ax+By+C=0$ are the two lines specified by the equations
$$\frac{ax+by+c}{\sqrt{a^2+b^2}}=\pm\frac{Ax+By+C}{\sqrt{A^2+B^2}}.$$

31. The Freedom-Equations of a Straight Line.

Let A (Fig. 25) be the point (a, c). Let $P(x, y)$ be any point on the locus specified by the equations $x=a+bt$, $y=c+dt$. Join AP. Let the parallel to $X'OX$ through A meet the parallel to $Y'OY$ through P in Q. Then $AQ=x-a$ and $QP=y-c$. But $x=a+bt$, or $x-a=bt$; and $y=c+dt$, or $y-c=dt$.

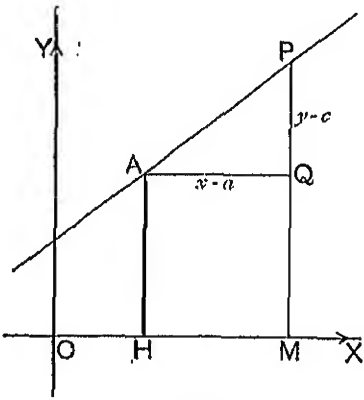


FIG. 25.

Hence $\qquad AQ=bt, \qquad QP=dt.$

Therefore, gradient of $AP=\dfrac{QP}{AQ}=\dfrac{dt}{bt}=\dfrac{d}{b}.$

Therefore the gradient of AP remains constant while P moves along the locus. Hence the locus of P is the straight line through $A(a, c)$ of gradient d/b.

That is, the freedom-equations
$$x=a+bt, \qquad y=c+dt$$
represent a straight line through the point (a, c) of gradient d/b.

Ex. 1. Make a table to show corresponding values of t, x, y when $x=2+4t$, $y=1+3t$. Draw a graph of x, y and find the constraint-equation connecting x, y.

t	-2	-1	0	1	2
x	-6	-2	2	6	10
y	-5	-2	1	4	7

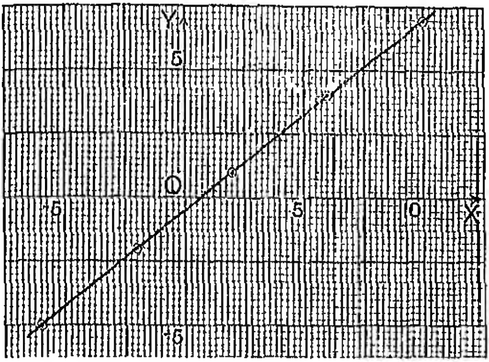


FIG. 26.

Corresponding values of x, y are plotted in Fig. 26; the graph of x, y is the straight line of Fig. 26. To find the constraint-equation, we have $\frac{x-2}{4}=t$ and $\frac{y-1}{3}=t$; hence $\frac{x-2}{4}=\frac{y-1}{3}$ or $3x-4y-2=0$.

Ex. 2. Prove that the straight line whose freedom-equations are $x=-2+3t$, $y=3-5t$ passes through the points $(-8, 13)$ and $(-1{\cdot}4, 2)$ referred to axes OX, OY; and find the corresponding values of t.

When $x=-8$, we have $-8=-2+3t$ or $t=-2$;
When $y=13$, we have $13=3-5t$ or $t=-2$;
When $x=-1{\cdot}4$, we have $-1{\cdot}4=-2+3t$ or $t=0{\cdot}2$;
When $y=2$, we have $2=3-5t$ or $t=0{\cdot}2$.

Ex. 3. Find the equations of the parallel and perpendicular to $x=-2+3t$, $y=-5-4t$ through the point $x=2$, $y=-3$.

The gradient of the given line is $\frac{-4}{3}$ or $-\frac{4}{3}$, by above section.

Hence the required parallel is $(y+3)=-\frac{4}{3}(x-2)$ or $4x+3y+1=0$; and the required perpendicular is $(y+3)=\frac{3}{4}(x-2)$ or $3x-4y-18=0$.

EXERCISES XI.

1. Prove that the straight line whose freedom-equations are $x=1+t$, $y=3+2t$ passes through the point $(1, 3)$ and has gradient 2. Draw the line.

2. Write down freedom-equations for the straight line which passes through $(3, 5)$ and has gradient $\frac{1}{2}$. Draw the line.

3. Prove that $(-1, -1)$ is a point on the straight line

$$x=2+3t, \quad y=1+2t.$$

Draw the graph of the line.

4. Prove that the straight line

$$x=3+2t, \quad y=5-t$$

is parallel to the straight line

$$x=2-4u, \quad y=1+2u.$$

5. Calculate the coordinates of the point of intersection of $x=-3+2t$, $y=2+3t$ and $x=4+5u$, $y=1-u$. Graph the two straight lines.

[We require $-3+2t=4+5u$ and $2+3t=1-u$ simultaneously. From the corresponding values of t (or u), calculate x, y from the given equations.]

6. Find the point of intersection of

$$x=2-3t,\ y=1+t\ ;\ \text{and}\ x=1-2u,\ y=2+3u.$$

7. Prove that the three straight lines, specified by the following equations, are concurrent, and find their point of intersection :

$$x=2+t, \quad y=8+2t\ ; \qquad \ldots\ldots(1)$$
$$x=1+u, \quad y=3+\tfrac{1}{2}u\ ; \qquad \ldots\ldots(2)$$
$$x=-4-3v, \quad y=5+3v. \qquad \ldots\ldots(3)$$

8. Find the constraint-equation of the line represented by

$$x=4+3t, \quad y=-3+7t.$$

[From first equation, $t=\frac{x-4}{3}$; from second, $t=\frac{y+3}{7}$. Hence $\frac{x-4}{3}=\frac{y+3}{7}$, *i.e.* $7x-3y=37$ is equation required.]

9. Find the constraint-equation of the straight line given by

$$x=-2+3t, \quad y=1-t.$$

10. Find the constraint-equation of the straight line

$$x=a+bt, \quad y=c+dt.$$

11. Prove that the straight lines

$$x=a+bt, \quad y=c+dt$$

and
$$x=a'+b'u, \quad y=c'+d'u$$

are (1) parallel if $d/b=d'/b'$; (2) perpendicular if $bb'+dd'=0$.

12. Find freedom-equations for (1) the parallel, (2) the perpendicular through (1, 2) to $x=2-3t$, $y=3+2t$.

13. Prove that the straight lines

$$x=5-2t, \; y=4+3t; \qquad x=7+6u, \; y=-11+4u$$

are perpendicular, and find their point of intersection. Draw the graphs of the lines.

14. Prove that the parallel through (h, k) to

$$x=a+bt, \quad y=c+dt$$

may have its freedom-equations written in the form

$$x=h+bu, \quad y=k+du.$$

15. Prove that the freedom-equations of the perpendicular through (h, k) to

$$x=a+bt, \quad y=c+dt$$

may be written
$$x=h+du, \quad y=k-bu.$$

CHAPTER V.

THE STRAIGHT LINE (*Continued*).

32. Different Forms of the Linear Equation. The following six forms of the linear equation $Ax+By+C=0$ are important:

(1) $y=mx+c$; (2) $y-y_1=m(x-x_1)$;

(3) $\dfrac{y-y_1}{x-x_1}=\dfrac{y_2-y_1}{x_2-x_1}$; (4) $\dfrac{x}{a}+\dfrac{y}{b}=1$;

(5) $x\cos\alpha+y\sin\alpha=p$; (6) $\dfrac{x-x_1}{\cos\theta}=\dfrac{y-y_1}{\sin\theta}=r.$

(1), (2) and (3) have already been explained in §§ 24, 26 and 27 respectively.

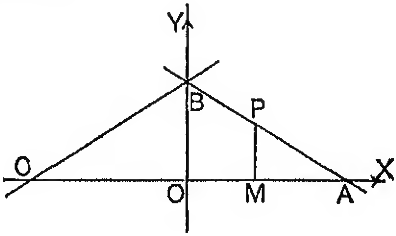


FIG. 27.

(4) **The equation** $\dfrac{x}{a}+\dfrac{y}{b}=1.$

Let a straight line not passing through the origin meet the x-axis at A and the y-axis at B. Let $OA=a$, $OB=b$.

Let $P(x, y)$ be any point on the line; then $\dfrac{x}{a}+\dfrac{y}{b}=1.$

Proof. Let M be the projection of P on the x-axis (Fig. 27).

Then Δs OAB, MAP are equiangular and therefore similar.

Therefore $\dfrac{MP}{OB}=\dfrac{MA}{OA}$, for all positions of P, where MP, OB, MA, OA are all steps.

Hence $$\frac{MP}{OB}=\frac{OA-OM}{OA},$$

that is, $$\frac{y}{b}=\frac{a-x}{a},$$

or $$\frac{x}{a}+\frac{y}{b}=1.$$

Ex. 1. If A, B are the points (2, 0) and (0, 3), find the equation of AB.

Here $a=2$, $b=3$; hence $\frac{x}{a}+\frac{y}{b}=1$ becomes $\frac{x}{2}+\frac{y}{3}=1$ or $3x+2y=6$, which is the equation of AB.

Ex. 2. If A, B are the points $(-2, 0)$ and (0, 3), find the equation of AB.

Here $a=-2$, $b=3$. Hence the required equation is $\frac{x}{-2}+\frac{y}{3}=1$, or $3x-2y+6=0$.

Ex. 3. Find the equation of the line joining $(-3, 0)$ and $(0, -5)$.

Ex. 4. Find the equation of the line joining (2, 0) and $(0, -4)$.

Ex. 5. Find, in sign and magnitude, the intercepts on the axes of x and y made by the lines

(i) $2x+3y=1$; (ii) $3x-2y=4$; (iii) $5x+7y+4=0$; (iv) $3x-5y+4=0$.

(i) $2x+3y=1$ may be written $\frac{x}{\frac{1}{2}}+\frac{y}{\frac{1}{3}}=1$. The intercepts are $\frac{1}{2}$ on OX and $\frac{1}{3}$ on OY. (Otherwise, put $y=0$ in $2x+3y=1$, then $2x=1$ or $x=\frac{1}{2}$; again put $x=0$ in $2x+3y=1$, then $3y=1$ or $y=\frac{1}{3}$.)

33. The Equation $x\cos\alpha+y\sin\alpha=p$. Let a straight line cut the x-axis at A and the y-axis at B (Fig. 28); let N be the foot of the perpendicular from the origin O on AB; let angle $XON=\alpha$ and $ON=p$. If $P(x, y)$ is any point on the line,

then $$x\cos\alpha+y\sin\alpha=p.$$

Let N be the point (x_1, y_1); then $x_1 = p\cos\alpha$, $y_1 = p\sin\alpha$.

Hence the gradient of PN (or AB) $= \dfrac{y-y_1}{x-x_1}$

$$= \frac{y-p\sin\alpha}{x-p\cos\alpha}. \quad \ldots\ldots(1)$$

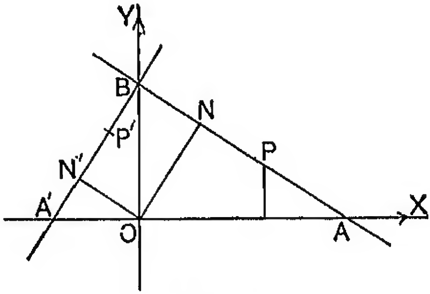


FIG. 28.

But the gradient of $ON = \tan\alpha$

$$= \frac{\sin\alpha}{\cos\alpha};$$

therefore the gradient of $PN = -\dfrac{\cos\alpha}{\sin\alpha}$. ....(2)

From (1) and (2), we get

$$\frac{y-p\sin\alpha}{x-p\cos\alpha} = -\frac{\cos\alpha}{\sin\alpha};$$

therefore $\quad x\cos\alpha + y\sin\alpha = p(\sin^2\alpha + \cos^2\alpha)$,

or $\quad x\cos\alpha + y\sin\alpha = p$,

since $\quad \sin^2\alpha + \cos^2\alpha = 1$.

COR. 1. If $ax+by=c$ (c positive) be put in the form $x\cos\alpha + y\sin\alpha = p$, then $p = \dfrac{c}{\sqrt{a^2+b^2}}$ and $\tan\alpha = \dfrac{b}{a}$.

For if $\tan\alpha = \dfrac{b}{a}$, we may put $\cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$ and $\sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$.

But $$ax+by=c.$$

Divide both sides by $\sqrt{a^2+b^2}$; then

$$x\cdot\frac{a}{\sqrt{a^2+b^2}}+y\cdot\frac{b}{\sqrt{a^2+b^2}}=\frac{c}{\sqrt{a^2+b^2}}. \qquad\text{.........(3)}$$

Therefore $$x\cos\alpha+y\sin\alpha=\frac{c}{\sqrt{a^2+b^2}};$$

or $$x\cos\alpha+y\sin\alpha=p,$$

if $$p=\frac{c}{\sqrt{a^2+b^2}}.$$

COR. 2. The length of the perpendicular from (x_1, y_1) to
$$x\cos\alpha+y\sin\alpha-p=0$$
is $$x_1\cos\alpha+y_1\sin\alpha-p.$$

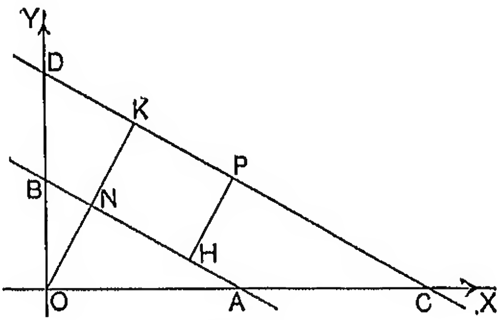


FIG. 29.

First, let us find the equation of the parallel through (x_1, y_1) to
$$x\cos\alpha+y\sin\alpha-p=0,$$
that is, the parallel CD through $P(x_1, y_1)$ to AB in Fig. 29.

By § 28, the equation of CD is
$$(x-x_1)\cos\alpha+(y-y_1)\sin\alpha=0$$
or $$x\cos\alpha+y\sin\alpha=x_1\cos\alpha+y_1\sin\alpha.$$

Hence OK = length of perpendicular from the origin to CD
$$=x_1\cos\alpha+y_1\sin\alpha.$$

But ON = length of perpendicular from the origin to AB

$= p$;

therefore HP = length of perpendicular from (x_1, y_1) to AB

$= NK$

$= OK - ON$

$= x_1 \cos\alpha + y_1 \sin\alpha - p.$

COR. 3. Going back to equation (3) of Cor. 1 we see that the length of the perpendicular from (x_1, y_1) to $ax+by+c=0$ is

$$\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}.$$

Ex. 1. Write the equation $3x+4y=7$ in the form

$$x\cos\alpha+y\sin\alpha=p.$$

Let $\tan\alpha=\frac{4}{3}$, so that we may write $\sin\alpha=\frac{4}{5}$ and $\cos\alpha=\frac{3}{5}$.

Now divide both sides of $3x+4y=7$ by 5.

We get
$$x \cdot \tfrac{3}{5} + y \cdot \tfrac{4}{5} = \tfrac{7}{5}$$
or
$$x\cos\alpha+y\sin\alpha=\tfrac{7}{5}.$$

Ex. 2. Find the length of the perpendicular from the origin to the line $4x+3y=8$.

Putting $\tan\alpha=\frac{3}{4}$, and proceeding as in Ex. 1, we write the equation $4x+3y=8$ in the form
$$x\cos\alpha+y\sin\alpha=\tfrac{8}{5},$$
where $\tan\alpha=\frac{3}{4}$. Hence the length of the perpendicular from the origin to $4x+3y=8$ is $\frac{8}{5}$. (Cp. § 30.)

Ex. 3. Write the equation $5x+12y=8$ in the form

$$x\cos\alpha+y\sin\alpha=p.$$

Ex. 4. Write each of the following equations in the form $x\cos\alpha+y\sin\alpha=p$; state the values of $\tan\alpha$ and p.

(i) $5x+12y=13$; (ii) $15x+20y=54$; (iii) $3x-4y=7$;
(iv) $12x-5y=10$; (v) $2x+3y=4$; (vi) $3x+y+2=0$;
(vii) $px+qy=r$; (viii) $lx+my+n=0$.

34. The equations $\dfrac{x-x_1}{\cos\theta}=\dfrac{y-y_1}{\sin\theta}=r.$

Let A be the point (x_1, y_1) (Figs. 15, 16, pp. 28, 29); through A draw the straight line of gradient $\tan\theta$; then if B

is any point (x, y) on the line and $AB=r$, we have

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r.$$

Let H, K be the projections of A, B on $X'OX$; let C be the projection of B on the parallel to $X'OX$ through A.

Then $$\cos\theta=\frac{AC}{AB}=\frac{HK}{AB}=\frac{OK-OH}{AB}=\frac{x-x_1}{r}.$$

Therefore $$\frac{x-x_1}{\cos\theta}=r. \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Similarly, $$\sin\theta=\frac{CB}{AB}=\frac{KB-KC}{AB}=\frac{KB-HA}{AB}=\frac{y-y_1}{r}.$$

Therefore $$\frac{y-y_1}{\sin\theta}=r. \quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From (1) and (2), we have

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r.$$

Ex. 1. Through the point $A(3, 1)$ is drawn the straight line making an angle of 45° with the x-axis. It meets the line $x+y=10$ at B; find the length of AB.

Ex. 2. A, B are the points $(a, 0)$, $(b, 0)$, and P is the point above the x-axis such that the triangle PAB is equilateral; find the coordinates of P.

35. Angle between Two Lines. If the equations of two lines are given, the lines could be graphed and the angle between them measured. Hence from the equations it is possible by calculation to find the angle between the lines represented by the equations. The rule is as follows:

If m_1 and m_2 are the gradients of two straight lines, and θ is the angle between the lines, then

$$\tan\theta=\frac{\mathrm{m}_1-\mathrm{m}_2}{1+\mathrm{m}_1\mathrm{m}_2},$$

θ being measured from the positive direction (§ 9) of the second line to the positive direction of the first line.

Let OP be the positive direction of the line of gradient m_1 (Fig. 30) and let angle $XOP=\theta_1$.

Let OQ be the positive direction of the line of gradient m_2, and let angle $XOQ=\theta_2$.

Then $\qquad \theta = Q\hat{O}P = X\hat{O}P - X\hat{O}Q = \theta_1 - \theta_2.$

Therefore $\quad \tan\theta = \tan(\theta_1-\theta_2) = \dfrac{\tan\theta_1 - \tan\theta_2}{1+\tan\theta_1\tan\theta_2}.$

But $\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$, so that

$$\tan\theta = \frac{m_1 - m_2}{1+m_1m_2}.$$

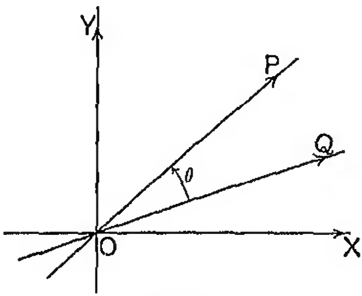


FIG. 30.

The numerical value of $(m_1-m_2)/(1+m_1m_2)$ is equal the tangent of the acute angle between the lines of gradient m_1 and m_2.

COR. If θ is the angle between the lines whose equations are $ax+by+c=0$ and $a'x+b'y+c'=0$,

then $$\tan\theta = \frac{ab'-a'b}{aa'+bb'}.$$

For, gradient of $ax+by+c=0$ is $-a/b=m_2$, say;
and $\quad$ gradient of $a'x+b'y+c'=0$ is $-a'/b'=m_1$, say.

Hence $$\tan\theta = \frac{m_1-m_2}{1+m_1m_2}$$

$$= \frac{-\dfrac{a'}{b'}+\dfrac{a}{b}}{1+\dfrac{a'}{b'}\cdot\dfrac{a}{b}} = \frac{ab'-a'b}{aa'+bb'}.$$

Ex. 1. Find the tangent of the angle between the lines $y=2x-3$ and $y=x+5$. With the help of tables, find the angle.

Here $m_1=2$ and $m_2=1$; therefore $\tan\theta=\frac{m_1-m_2}{1+m_1m_2}=\frac{2-1}{1+2.1}=\frac{1}{3}$. From the tables, $\theta=18^\circ\ 26'$.

Ex. 2. Find the tangent of the angle between the lines $2x-y=3$ and $3x+4y=5$. Also find the acute angle between the lines.

Put $m_1=$gradient of $2x-y=3$, so that $m_1=2$; and $m_2=$gradient of $3x+4y=5$, so that $m_2=-\frac{3}{4}$.

Then $$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}=\frac{2+\frac{3}{4}}{1-2.\frac{3}{4}}=-\frac{11}{2}.$$

The acute angle is roughly $79^\circ\ 42'$.

Ex. 3. Prove that the two lines $y=2x+4$ and $y=3x+4$ are inclined at the same angle as $x-y+2=0$ and $3x-4y+1=0$.

Let $\theta_1=$angle between lines $y=2x+4$ and $y=3x+4$.

Then $$\tan\theta_1=\frac{m_1-m_2}{1+m_1m_2}=\frac{2-3}{1+2.3}=-\frac{1}{7}.$$

Let $\theta_2=$angle between lines $x-y+2=0$ and $3x-4y+1=0$.

Then $$\tan\theta_2=\frac{m_1-m_2}{1+m_1m_2}=\frac{1-\frac{3}{4}}{1+1.\frac{3}{4}}=\frac{1}{7}.$$

Hence $\tan\theta_1=\tan\theta_2$, numerically; that is, the first pair of lines is inclined at the same angle as the second pair.

Ex. 4. If $P(x, y)$ is a point above the axis of x, and A and B are the points $(1, 0)$ and $(-1, 0)$ respectively, prove that
$$x^2+y^2=2y+1,$$
if the angle APB is half a right angle.

Let $m_1=$gradient of $PA=\frac{y}{x-1}$.

Let $m_2=$gradient of $PB=\frac{y}{x+1}$.

If θ is the angle measured from the positive direction of PB to the positive direction of PA, then $\tan\theta=1$.

Also $$\tan\theta=\frac{m_1-m_2}{1+m_1m_2};$$

$$\therefore\ 1=\frac{\frac{y}{x-1}-\frac{y}{x+1}}{1+\frac{y}{x-1}\cdot\frac{y}{x+1}};$$

this reduces to $$x^2+y^2=2y+1.$$

Ex. 5. If $P(x, y)$ is below the axis of x, and A and B are the points $(1, 0)$ and $(-1, 0)$ respectively, prove that

$$x^2+y^2+2y=1,$$

if the angle APB is half a right angle.

Ex. 6. If A, B, P have coordinates $(1, 0)$, $(-1, 0)$ and (x, y) respectively, and if the angle APB is a right angle, prove that $x^2+y^2=1$.

Ex. 7. From the formula $\tan\theta=\dfrac{m_1-m_2}{1+m_1m_2}$ deduce the condition that two lines of gradients m_1 and m_2 should be parallel, and also the condition that they should be perpendicular.

Ex. 8. Find the tangents of the angles of the triangle whose sides are the lines $2x-3y-5=0$, $5x-y-3=0$, and $x+y+5=0$. Also calculate the angles of the triangle.

Ex. 9. Find the equations of the lines through $(2, 7)$ which are inclined at an angle of 45° to the line $x+2y=4$.

Ex. 10. Find the equations of the sides of an isosceles triangle whose vertex is the point (a, b), whose base is the line $lx+my+n=0$, and each of whose base angles is α.

Ex. 11. If the angle measured from the positive direction of the line $2x+3y=7$ to the positive direction of a line passing through $(3, 7)$ is 45°, find the equation of this line.

Ex. 12. If the angle measured from the positive direction of the line $y=mx+c$ to the positive direction of a line through (x_1, y_1) is α, prove that the equation of this line is

$$y-y_1=\frac{m+\tan\alpha}{1-m\tan\alpha}(x-x_1).$$

36. Bisectors of Angles between Two Lines. If any two intersecting straight lines are drawn, then two angles are formed, each of which has a bisector. We know from elementary geometry that any point on either bisector is equidistant from the two lines.

Let $\quad ax+by+c=0 \ldots(1), \quad Ax+By+C=0 \ldots(2)$

be two intersecting straight lines; it is required to form the equations of the bisectors of the angles between them.

If (x_1, y_1) is a point on either bisector, then

length of perpendicular from (x_1, y_1) to line (1)

$=$ length of perpendicular from (x_1, y_1) to line (2).

But length of perpendicular from (x_1, y_1) to $ax+by+c=0$

is, by § 30, $$\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}.$$

And length of perpendicular from (x_1, y_1) to $Ax+By+C=0$

is $$\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}.$$

Hence $$\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}=\pm\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}},$$

the double sign being necessary, since the left-hand and right-hand expressions may be either positive or negative.

For x_1, y_1 write x, y, and we obtain

$$\frac{\mathbf{ax+by+c}}{\sqrt{\mathbf{a^2+b^2}}}=\pm\frac{\mathbf{Ax+By+C}}{\sqrt{\mathbf{A^2+B^2}}}$$

as the equations of the two bisectors.

The reader should revise Exs. 13–22 on p. 55.

Ex. 1. Find the equations of the bisectors of the angles between the lines $2x-y+2=0$ and $x+2y+7=0$; and draw the four lines in the same diagram.

Length of perpendicular from (x, y) to first line $=\dfrac{2x-y+2}{\sqrt{5}}$;

and length of perpendicular from (x, y) to second line $=\dfrac{x+2y+7}{\sqrt{5}}$.

Hence equations of bisectors are

$$\frac{2x-y+2}{\sqrt{5}}=\pm\frac{x+2y+7}{\sqrt{5}},$$

that is, $$2x-y+2=\pm(x+2y+7),$$

or $$x-3y=5 \text{ and } 3x+y+9=0.$$

Ex. 2. Find the equations of the bisectors of the angles between the lines $7x-3y+1=0$ and $3x-7y+2=0$; and draw the four lines in the same diagram.

Ex. 3. Find the equations of the bisectors of the angles formed by the axes $X'OX$ and $Y'OY$.

Ex. 4. Find the equations of the bisectors of the angles between the lines $3x-4y+1=0$ and $5x+12y+4=0$.

Ex. 5. Find the equations of the bisectors of the angles formed by the lines $8x-15y+20=0$ and $5x+12y=20$.

Ex. 6. Find the intercepts made on the x-axis by the bisectors of the angles between the lines $15x+8y+30=0$ and $12x+5y=25$.

Ex. 7. Find the equations of the bisectors of the angles between the lines $3x-y+2=0$ and $2x+4y=7$.

Ex. 8. Prove that $x+y=1$ is equally inclined to the lines

$$4x-3y+2=0 \text{ and } 3x-4y+3=0,$$

and passes through their intersection.

Ex. 9. Prove that $12x-2y+7=0$ is equally inclined to the lines

$$5x-7y+4=0 \text{ and } 7x+5y+3=0.$$

37. Intersection of Lines: Concurrency. If two straight lines be specified by equations in x, y, referred to rectangular axes, then as far as the coordinates of their point of intersection are concerned, the two equations may be regarded as simultaneous equations. For example, if we solve the simultaneous equations

$$2x-y=7; \quad 4x+3y+1=0,$$

we find $x=2$, $y=-3$. This shows that $(2, -3)$ is the point of intersection of the lines specified by the equations $2x-y=7$ and $4x+3y+1=0$.

Again, if we have the three equations,

$$2x-3y=12, \quad \ldots\ldots(1)$$

$$3x+2y=5, \quad \ldots\ldots(2)$$

$$7x+10y=1, \quad \ldots\ldots(3)$$

and solve (1) and (2) as simultaneous equations, we find $x=3$, $y=-2$. If we now substitute $x=3$, $y=-2$ in $7x+10y$, the left of (3), we find $7x+10y=1$, so that the *three* equations are simultaneous equations for $x=3$, $y=-2$. This means that the three straight lines represented by equations (1), (2), and (3) are concurrent at the point $(3, -2)$.

More generally, the intersecting straight lines specified by the equations

$$ax+by+c=0 \text{ and } a'x+b'y+c'=0$$

meet at the point $\left(\dfrac{bc'-b'c}{ab'-a'b}, \dfrac{ca'-c'a}{ab'-a'b}\right)$.

Further, the three straight lines specified by the equations

$$a_1x+b_1y+c_1=0,$$
$$a_2x+b_2y+c_2=0,$$
$$a_3x+b_3y+c_3=0,$$

are concurrent if

$$a_1(b_2c_3-b_3c_2)+b_1(c_2a_3-c_3a_2)+c_1(a_2b_3-a_3b_2)=0.$$

Ex. 1. Find the point of intersection of the lines

$$x+y=3\ ;\ 2x-3y=11.$$

Ex. 2. Prove that the following equations represent three concurrent straight lines, and find their point of intersection:

$$3x+4y+2=0\ ;\ x-3y+5=0\ ;\ 2x+7y=3.$$

Ex. 3. Draw the straight lines $y=mx+2$, when $m=1$, $m=-2$, $m=3$. Find their common point.

Ex. 4. Draw the lines $y-3=m(x-2)$, when $m=1, m=-1, m=-2$. Prove that these three lines are concurrent, and find their common point.

Ex. 5. Draw the two lines represented by

$$(2x+y-7)+k(3x-4y-5)=0,$$

when $k=2$ and $k=-3$. Find their point of intersection.

Ex. 6. Draw the three lines specified by

$$2x-3y-12+k(7x+10y-1)=0,$$

when $k=1$, $k=-2$, $k=3$. Prove that they are concurrent, and find their point of intersection.

Ex. 7. Prove that the two straight lines

$$(2x-y-1)+p(x-y+1)=0$$

and

$$(2x-y-1)+q(x-y+1)=0$$

intersect at the point (2, 3).

Ex. 8. Prove that the two straight lines

$$(x+y-1)+p(4x+3y-6)=0$$

and

$$(x+y-1)+q(4x+3y-6)=0$$

intersect where $x+y-1=0$ and $4x+3y-6=0$ intersect.

Ex. 9. Prove that the straight lines obtained from the equation

$$(x-y-a+b)+k(x+y-a-b)=0,$$

by giving k various values, will pass through the same point, and find the coordinates of the point.

Put $k=p$, then $k=q$. We obtain

$$(x-y-a+b)+p(x+y-a-b)=0 \quad\ldots\ldots\ldots\ldots(1)$$

and
$$(x-y-a+b)+q(x+y-a-b)=0. \quad\ldots\ldots\ldots\ldots(2)$$

To solve these equations, first subtract:

then
$$p(x+y-a-b)-q(x+y-a-b)=0,$$

that is,
$$(p-q)(x+y-a-b)=0$$

or
$$x+y-a-b=0. \quad\ldots\ldots\ldots\ldots(3)$$

Now substitute from (3) in (1), and obtain

$$x-y-a+b=0. \quad\ldots\ldots\ldots\ldots(4)$$

We have now to solve (3) and (4).

Add: then $\qquad 2x-2a=0$ or $x=a$.

Subtract: then $\qquad 2y-2b=0$ or $y=b$.

Hence, whatever value k has in the given equation, all the lines represented by the equation pass through the same point (a, b).

Ex. 10. Prove that all the lines represented by the equation

$$(x-y+a-b)+k(x+y-a-b)=0,$$

by giving k various values, pass through a fixed point, and find its coordinates.

Ex. 11. If the two straight lines $ax+by+c=0$ and $a'x+b'y+c'=0$ meet at the point (p, q), show that (p, q) lies on all the lines represented by

$$ax+by+c+k(a'x+b'y+c')=0,$$

where k is a varying constant.

Ex. 12. Find where the lines $2x+y-3=0$ and $x-3y+2=0$ meet. Prove that all the lines represented by

$$(2x+y-3)+k(x-3y+2)=0,$$

where k is a varying constant, pass through the point.

Ex. 13. Prove that, if k is a varying constant, the system of lines $(3x-2y+4)+k(2x+4y-5)=0$ passes through the intersection of $3x-2y+4=0$ and $2x+4y-5=0$.

Ex. 14. Interpret geometrically the equation

$$(2x-y-3)+k(x+3y+2)=0,$$

where k is a varying constant.

38. System of Concurrent Lines. If two intersecting straight lines are specified by the equations

$$ax+by+c=0 \quad\text{and}\quad a'x+b'y+c'=0,$$

then all straight lines through their point of intersection are specified by the equation

$$ax+by+c+k(a'x+b'y+c')=0,$$

where k is a varying constant (parameter). The parameter k is constant for any one line of the system, but varies as the line varies.

Proof. Let the straight lines

$$ax+by+c=0 \quad \text{and} \quad a'x+b'y+c'=0$$

meet at the point (p, q), so that

$$ap+bq+c=0 \text{ and } a'p+b'q+c'=0.$$

$\left(\text{Of course, } p \text{ stands for } \dfrac{bc'-b'c}{ab'-a'b} \text{ and } q \text{ for } \dfrac{ca'-c'a}{ab'-a'b}.\right)$

Then, whatever constant value k has,

$$ap+bq+c+k(a'p+b'q+c')=0.$$

Therefore, whatever constant value k has, the line

$$ax+by+c+k(a'x+b'y+c')=0$$

passes through the point (p, q), that is, through the intersection of the lines

$$ax+by+c=0 \quad \text{and} \quad a'x+b'y+c'=0.$$

Again, let the three equations

$$a_1x+b_1y+c_1=0, \quad \text{.....................(1)}$$
$$a_2x+b_2y+c_2=0, \quad \text{.....................(2)}$$
$$a_3x+b_3y+c_3=0 \quad \text{.....................(3)}$$

be such that

$$l(a_1x+b_1y+c_1)+m(a_2x+b_2y+c_2)+n(a_3x+b_3y+c_3)=0 \text{...(4)}$$

for all values of x and y, where l, m, n are constants other than zero; then the three straight lines (supposed not parallel) represented by (1), (2), (3) are concurrent. For, let the lines (1) and (2) meet at the point (p, q), so that p stands for $\dfrac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}$ and q for $\dfrac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}$.

Then $a_1p+b_1q+c_1=0$ and $a_2p+b_2q+c_2=0.$

But, by (4),

$$l(a_1p+b_1q+c_1)+m(a_2p+b_2q+c_2)+n(a_3p+b_3q+c_3)=0.$$

Therefore $$n(a_3p+b_3q+c_3)=0,$$
that is, $$a_3p+b_3q+c_3=0, \text{ since } n \neq 0.$$

Hence the line $a_3x+b_3y+c_3=0$ passes through the point (p, q), that is, through the intersection of lines (1) and (2). Hence the lines (1), (2), (3) are concurrent.

Ex. 1. Find the equation of the line joining the origin to the point of intersection of $3x-y+2=0$ and $2x+y+3=0$.

The equation $3x-y+2+k(2x+y+3)=0$, for varying values of the constant k, represents all straight lines through the intersection of $3x-y+2=0$ and $2x+y+3=0$. It remains to find the particular value of k which gives the line through the origin. Put $x=0$, $y=0$ in the equation containing k; then

$$2+3k=0;$$
$$\therefore k=-\tfrac{2}{3}.$$

Hence $$3x-y+2-\tfrac{2}{3}(2x+y+3)=0,$$
that is, $$5x-5y=0 \text{ or } x-y=0$$
is the line required.

Ex. 2. Find the equation of the straight line joining $(3, -2)$ and the point of intersection of $3x+5y-7=0$ and $2x-3y+1=0$.

Let the equation be

$$3x+5y-7+k(2x-3y+1)=0. \quad\text{..........................(1)}$$

The point $(3, -2)$ lies on the line.

Therefore $$3.3+5(-2)-7+k\{2.3-3(-2)+1\}=0,$$
that is, $$-8+13k=0$$
or $$k=\tfrac{8}{13}.$$

Going back to (1), we see that the required equation is

$$3x+5y-7+\tfrac{8}{13}(2x-3y+1)=0,$$
or $$55x+41y-83=0.$$

Ex. 3. Find the equation of the line joining the origin to the point of intersection of $5x-7y+2=0$ and $2x+8y=11$.

Ex. 4. Find the equations of the lines joining the following points to the intersections of the following lines:

(i) Point: $(1, -1)$; lines: $x-y+1=0$; $2x-y=1$;
(ii) Point: $(0, 0)$; lines: $3x+4y=7$; $2x-5y=8$;
(iii) Point: $(3, 2)$; lines: $3x-5y=12$; $x+y+3=0$.

Ex. 5. Find the equation of the parallel to $2x-3y+1=0$ through the intersection of $x+y+1=0$ and $2x-y+5=0$.

Let the line be $$x+y+1+k(2x-y+5)=0. \quad\text{..........................(1)}$$

Then gradient of the line $=\dfrac{1+2k}{k-1}$.

But gradient of $2x-3y+1=0$ is $\frac{2}{3}$.

Hence $$\frac{1+2k}{k-1}=\frac{2}{3},$$

that is, $$k=-\tfrac{5}{4}.$$

Going back to equation (1), we obtain

$$x+y+1-\tfrac{5}{4}(2x-y+5)=0,$$

that is, $$6x-9y+21=0$$

or $$2x-3y+7=0.$$

Ex. 6. Find the equation of the line drawn through the intersection of the lines $3x+4y=7$ and $4x-5y+11=0$, (i) to pass through the point $(-2, -5)$; (ii) parallel to $3x-2y=1$; (iii) perpendicular to $3x-2y=1$.

Ex. 7. Straight lines are drawn through the vertices of the triangle formed by the lines $2x-y+1=0$, $3x+2y=4$, $x-y=2$, (i) parallel to the opposite sides; (ii) perpendicular to the opposite sides. Find the equations of the two systems of lines.

EXERCISES XII.

1. Find the intercepts made on the axes by the line $2x-3y=4$.

2. Find the intercepts made on the axes by the line joining the points (x_1, y_1) and (x_2, y_2).

3. Find the intercepts made on the axes by the line whose freedom equations are $x=a+bt$, $y=c+dt$.

4. Find the coordinates of the point of intersection of the two lines $x=a+bt$, $y=c+dt$ and $x=a'+b'u$, $y=c'+d'u$.

5. Find $\tan\alpha$ and p if the line $ax+by+c=0$ be expressed in the form $x\cos\alpha+y\sin\alpha=p$.

6. Prove that the line $x-5y+6=0$ passes through the point of intersection of the lines $3x-2y+2=0$ and $2x+3y-4=0$, and bisects the angle between them.

7. Prove that the equation $(x-x_1)(y-y_2)=(x-x_2)(y-y_1)$ represents the straight line joining (x_1, y_1) and (x_2, y_2).

8. Prove that the equation

$$C(ax+by+c)=c(Ax+By+C)$$

represents the line joining the origin to the point of intersection of $ax+by+c=0$ and $Ax+By+C=0$.

9. Represent graphically the lines of the system

$$y-2x+ky-3+k=0,$$

for $k=1, -2, 3, -4$. Show that all the lines pass through a common point.

10. Find the conditions that the two systems of equations

$$x=a+bt, \quad y=c+dt \quad \text{and} \quad x=a'+b'u, \quad y=c'+d'u$$

should represent (i) parallel lines; (ii) perpendicular lines; (iii) [illegible] and the same straight line.

11. Find the condition that the lines $x=a+bt$, $y=c+dt$ and $Ax+By+C=0$ should be (i) parallel; (ii) perpendicular.

12. Find the point of intersection of the intersecting lines

$$x=a+bt, \quad y=c+dt \quad \text{and} \quad Ax+By+C=0.$$

13. One line of the system

$$2x-y+4+k(x+4y+5)=0$$

meets the x-axis at A and the y-axis at B. If $OA=-2$, find OB.

14. Use a book of Tables to find the angle formed by the [illegible] $4x-2y+3=0$ and $2x-4y+7=0$. What are the equations of the bisectors of the angles so formed?

15. If b is positive, prove that ax_1+by_1+c is positive or [illegible] according as (x_1, y_1) is above or below the line $ax+by+c=0$.

16. Is the origin above or below the line $x+2y=3$?

17. Prove that the origin and the point $(-1, -2)$ are on [illegible] sides of the line $2x+3y+5=0$. Find the point where the [illegible] $(0, 0)$ to $(-1, -2)$ meets $2x+3y+5=0$.

18. If b is positive, prove that the point $(a-b, a+b)$ lies above [illegible] line $ax+by+1=0$.

19. Prove that the origin lies below the line $a^2x+b^2y=c^2$.

20. Prove that the origin lies within that angle, formed by [illegible] lines $2x-y+3=0$ and $x-2y+5=0$, in which lies the bisector $x+y=$ [illegible].

21. Prove that the three straight lines

$$x+y=2,$$
$$(b-c)x+(c-a)y+(a-b)=0,$$
$$a(b-c)x+b(c-a)y+c(a-b)=0$$

are concurrent.

22. Verify that

$$2(2x-y+1)-3(x+y-1)-(x-5y+5)=0$$

for all values of x and y; and deduce a property of the three lines

$$2x-y+1=0, \quad x+y=1, \quad x+5=5y.$$

23. Find the equation of the line joining the intersection of $5x-3y+4=0$ and $x+7y-2=0$ to the intersection of [illegible] and $3x+2y=0$.

24. The equations $y=ax+b$, $y=cx+d$ represent two lines. [illegible] c in terms of a, (i) when the lines are parallel; (ii) when the lines [illegible] perpendicular; (iii) when the lines form an angle of $45°$.

25. Find the equation of the straight line OP which passes through the origin and also through P, the point of intersection of $4x-2y=3$ and $5y-3x=4$. Write down the equation of the straight line through the origin at right angles to OP; and find the coordinates of the points in which it intersects the given lines.

26. Prove that the lines of the system $y=(1-k)x+(1+k)$ all pass through a fixed point, and find the coordinates of the point.

27. If, in an isosceles triangle ABC, the angle at A is a right angle, and if $(2, 7)$, $(6, 1)$ are the coordinates of A and B, find the coordinates of each possible position of C.

28. The coordinates of A, B, P are $(a, 0)$, $(b, 0)$, (x, y). Prove that the tangent of the angle APB is

$$\pm\frac{(a-b)y}{(x-a)(x-b)+y^2}.$$

29. Find the condition that the three lines

$$ax+by+c=0,$$
$$bx+cy+a=0,$$
$$cx+ay+b=0$$

meet in a point.

30. Perpendiculars are drawn from the origin to the straight lines whose equations are $x+2y=3$ and $2x+3y=5$; find the equation of the straight line which joins the feet of the perpendiculars.

31. Find the tangent of the angle between the lines joining (x, y) to (a, b), (c, d).

32. Find what value p must have in order that the straight lines $px+4y=6$, $3x+4y=5$, $2x+3y=4$ may meet in a point.

33. A, B are the points $(9, 0)$, $(0, 12)$. Find the coordinates of the point of intersection of the medians of the triangle OAB, where O is the origin.

CHAPTER VI.

PAIRS OF STRAIGHT LINES. HARMONIC RANGES AND PENCILS. CHANGE OF ORIGIN AND AXES.

39. The Equation uv=0. Certain equations in x and y of higher degree than the first can be graphically represented by means of two or more straight lines.

For example, consider the equation

$$12x^2-2xy-10y^2-6x+17y-6=0. \qquad \text{(i)}$$

Put $x=-2$ in (i);

then $\qquad 48+4y-10y^2+12+17y-6=0,$

which reduces to $\qquad 10y^2-21y-54=0,$

that is, $\qquad (2y+3)(5y-18)=0,$

so that $\qquad y=-1\cdot5$ or $3\cdot6$.

Hence $x=-2$, $y=-1\cdot5$ and $x=-2$, $y=3\cdot6$ are points on the graph of (i). To obtain an idea of the graph of (i), construct a table as follows, noting that to each value assigned to x in (i) there correspond two values of y:

x	-2	-1	0	1	2
y	$-1\cdot5$ or $3\cdot6$	$-0\cdot5$ or $2\cdot4$	$0\cdot5$ or $1\cdot2$	$1\cdot5$ or 0	$2\cdot5$ or $-1\cdot2$

giving A, A' B, B' C, C' D, D' E, E' of Fig. [illegible].

Since the points A, B, C, D, E lie on one straight line and the points A', B', C', D', E' lie on a second straight line, it is possible that all the point-pairs belonging to equation (i) also belong to one or other of these lines. It is easy to show that this is true.

For $\qquad 12x^2-2xy-10y^2-6x+17y-6$

can be factorised and expressed as a product thus:

$$(2x-2y+1)(6x+5y-6).$$

If then (x, y) is any point on the graph of (i), it follows that

$$(2x-2y+1)(6x+5y-6)=0.$$

Hence $$2x-2y+1=0, \quad\text{.................(ii)}$$

or $$6x+5y-6=0, \quad\text{.................(iii)}$$

or *both* $2x-2y+1=0$ *and* $6x+5y-6=0$. ....(iv)

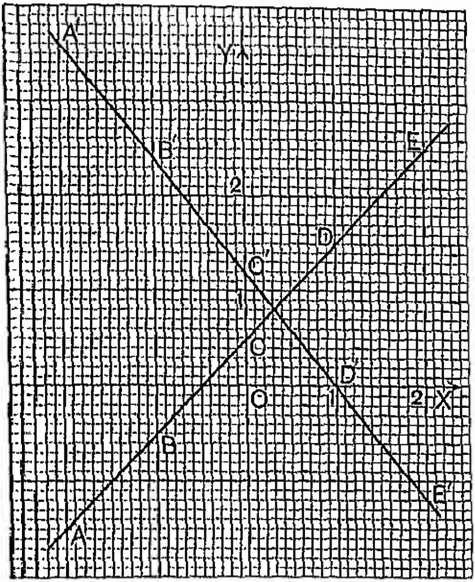


FIG. 31.

If (ii) is true, (x, y) must lie on the straight line AE of Fig. 31; if (iii) is true, (x, y) must lie on the straight line $A'E'$; if (iv) is true, (x, y) must be the point of intersection of AE and $A'E'$. Hence the graph of equation (i) is completely represented by two straight lines.

More generally, if $u=0$ and $v=0$ are linear equations in x and y, then the equation

$$uv=0$$

represents the two straight lines $u=0$ and $v=0$.

If $u=0$, $v=0$, $w=0$ are linear equations in x and y, then the equation $$uvw=0$$

represents the three straight lines $u=0$, $v=0$, $w=0$.

Ex. 1. Find an equation which represents the lines

$$x-2y+1=0 \text{ (i)} \quad \text{and} \quad 2x+3y+1=0 \text{ (ii)}.$$

Multiplying $x-2y+1$ by $2x+3y+1$, we obtain

$$2x^2-xy-6y^2+3x+y+1.$$

The equation $2x^2-xy-6y^2+3x+y+1=0$ represents the two lines (i) and (ii).

Ex. 2. Factorise $2x^2-xy-y^2+4x+5y-6$; and find the two straight lines represented by

$$2x^2-xy-y^2+4x+5y-6=0. \qquad \text{.......................(A)}$$

First Step: Factorise $2x^2-xy-y^2$; we get $(2x+y)(x-y)$.

Second Step: Form the product $(2x+y+m)(x-y+n)$.

Third Step: Multiply out:

$$2x^2-xy-y^2+(m+2n)x+(-m+n)y+mn.$$

Fourth Step: Comparing the coefficients of x and y, and also the absolute term in this expression with those in the given expression, put

$$m+2n=4 \text{ (i)}, \quad -m+n=5 \text{ (ii)}, \quad mn=-6 \text{ (iii)}.$$

Fifth Step: Solve (i) and (ii) as simultaneous equations in m and n, getting $m=-2$, $n=3$; and verify that (iii) is then satisfied.*

We find $2x^2-xy-y^2+4x+5y-6=(2x+y-2)(x-y+3)$.

The two straight lines represented by (A) are $2x+y-2=0$ and $x-y+3=0$.

Ex. 3. Factorise $6x^2+13xy+5y^2-16x-22y+8$.

Ex. 4. Factorise $77x^2-65xy-88y^2-52x+107y-9$.

Ex. 5. Show that the following equations represent two straight lines, and find their equations:

(i) $(x+y-1)(x-y+1)=0$; (ii) $x^2-y^2+x-y=0$;

(iii) $2x^2-xy-y^2-3x+1=0$; (iv) $6x^2+5xy-6y^2-3x+28y-30=0$;

(v) $15x^2+19xy-10y^2+7x+22y-4=0$;

(vi) $84x^2-66xy-54y^2+61x+93y-40=0$;

(vii) $3x^2-10xy+3y^2=0$; (viii) $abx^2-(a^2+b^2)xy+aby^2=0$;

(ix) $a(b-c)x^2+b(c-a)xy+c(a-b)y^2=0$; (x) $ax^2+2hxy+by^2=0$.

*If the last term in the given expression viz., -6, be replaced by any other number, then (iii) would not be satisfied for $m=-2$, $n=3$. This shows that it is exceptional for such expressions to have factors.

40. Condition that $ax^2+2hxy+by^2+2gx+2fy+c=0$ should represent two Straight Lines. The necessary and sufficient condition that the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{............(1)}$$

should represent two straight lines is that

$$abc+2fgh-af^2-bg^2-ch^2=0. \quad \text{..............(2)}$$

Proof. (i) If $a \neq 0$, instead of equation (1) we may consider the equation

$$a^2x^2+2ahxy+aby^2+2agx+2afy+ac=0; \quad \text{......(3)}$$

for (3) may be derived from (1) by multiplying both sides of (1) by a, and (1) from (3) by dividing both sides of (3) by a, since $a \neq 0$.

From (3), we have

$$a^2x^2+2ahxy+2agx=-aby^2-2afy-ac.$$

Add $h^2y^2+2ghy+g^2$ to both sides in order that the left side may be the square of $ax+hy+g$; then

$$a^2x^2+h^2y^2+g^2+2ahxy+2agx+2ghy$$
$$=(h^2-ab)y^2+2(gh-af)y+(g^2-ac),$$

that is,

$$(ax+hy+g)^2-\{(h^2-ab)y^2+2(gh-af)y+(g^2-ac)\}=0. \quad (4)$$

Now $(h^2-ab)y^2+2(gh-af)y+(g^2-ac)$ is a perfect square as regards y (that is, is the square of an expression of the first degree in y)

if
$$4(gh-af)^2=4(h^2-ab)(g^2-ac),$$

which reduces to

$$a^2bc+2afgh-a^2f^2-abg^2-ach^2=0$$

or, since $a \neq 0$, to

$$abc+2fgh-af^2-bg^2-ch^2=0.$$

This is the condition that the left-hand expression of (4) can be factorised. Hence it is the condition that equation (1) should represent two straight lines. Since the steps of the argument are reversible, the condition is both necessary and sufficient.

(ii) If $a=0$ and $b\neq 0$, by interchanging a and b, and x and y, we see that the same condition holds.

(iii) If $a=0$ and $b=0$, but $h\neq 0$, then (1) becomes

$$2hxy+2gx+2fy+c=0$$

or

$$xy+\frac{g}{h}\cdot x+\frac{f}{h}y+\frac{c}{2h}=0,$$

since $h\neq 0$.

Since the left-hand must factorise, there must be numbers, p and q say, such that

$$xy+\frac{g}{h}x+\frac{f}{h}y+\frac{c}{2h}=(x+p)(y+q)$$
$$=xy+qx+py+pq,$$

that is, such that

$$\frac{g}{h}=q,\quad \frac{f}{h}=p,\quad \frac{c}{2h}=pq.$$

Hence

$$\frac{c}{2h}=\frac{g}{h}\cdot\frac{f}{h}$$

or

$$2fgh-ch^2=0.$$

But this is what (2) becomes when $a=0$ and $b=0$. Hence condition (2) still holds.

(iv) If $a=0$, $b=0$, $h=0$, equation (1) becomes linear in x and y, and therefore drops out of the discussion.

41. The Equation $ax^2+2hxy+by^2=0$. Equation (1) of the preceding section, when $g=0$, $f=0$, $c=0$, takes the form

$$ax^2+2hxy+by^2=0,$$

which is of special interest. This equation always represents two straight lines through the origin; if h^2-ab is positive the lines can be graphed and are real, if h^2-ab is negative the lines cannot be graphed and are imaginary, if h^2-ab is zero the two lines consist of the same line twice over.

For example, the equation

$$3x^2+7xy-6y^2=0 \quad \text{..............................(1)}$$

can be written in the form

$$(3x-2y)(x+3y)=0,$$

and therefore represents the two lines

$$3x-2y=0 \quad \text{and} \quad x+3y=0,$$

that is, the lines through the origin of gradients $\frac{3}{2}$ and $-\frac{1}{3}$.

The equation $$x^2+xy+y^2=0 \quad \ldots\ldots\ldots\ldots (2)$$

can be written in the form

$$\left(x+\frac{1+i\sqrt{3}}{2}y\right)\left(x+\frac{1-i\sqrt{3}}{2}y\right)=0,$$

where $i=\sqrt{-1}$, and therefore represents the two lines

$$x+\frac{1+i\sqrt{3}}{2}y=0 \quad \text{and} \quad x+\frac{1-i\sqrt{3}}{2}y=0;$$

these are two imaginary lines through the origin.

The equation $$4x^2+12xy+9y^2=0 \quad \ldots\ldots\ldots\ldots (3)$$

can be written in the form

$$(2x+3y)(2x+3y)=0,$$

and therefore represents the line

$$2x+3y=0, \text{ twice.}$$

If the equation $ax^2+2hxy+by^2=0$

represents the two lines through the origin of gradients m_1 and m_2, then the lines are

$$y-m_1x=0 \quad \text{and} \quad y-m_2x=0.$$

These are both represented by

$$(y-m_1x)(y-m_2x)=0,$$

that is, $$y^2-(m_1+m_2)xy+m_1m_2x^2=0. \quad \ldots\ldots\ldots\ldots (\mathrm{I})$$

But $$ax^2+2hxy+by^2=0$$

can be written in the form

$$y^2+\frac{2h}{b}\cdot xy+\frac{a}{b}\cdot x^2=0. \quad \ldots\ldots\ldots\ldots (\mathrm{II})$$

Hence (I) and (II) must be the same equation. Therefore

$$m_1+m_2=-\frac{2h}{b} \quad \text{and} \quad m_1m_2=\frac{a}{b}. \quad \ldots\ldots\ldots\ldots (\mathrm{III})$$

These relations are important, as the following exercises will show.

Ex. 1. Find the condition that the two lines represented by the equation $ax^2+2hxy+by^2=0$ may be perpendicular to one another.

Let the gradients of the lines be m_1 and m_2.

Then either $$m_1=-\frac{1}{m_2} \quad\text{or}\quad m_2=-\frac{1}{m_1},$$

that is, $$m_1m_2+1=0.$$

Hence, from (III), $$\frac{a}{b}+1=0$$

or $$a+b=0.$$

Since the steps are reversible, the condition is sufficient as well as necessary.

Ex. 2. Find the condition that the gradient of one of the lines represented by $ax^2+2hxy+by^2=0$ should be double that of the other.

Let the gradients be m_1 and m_2.

Then either $$m_1=2m_2 \quad\text{or}\quad m_2=2m_1,$$

that is, $$m_1-2m_2=0 \quad\text{or}\quad m_2-2m_1=0.$$

Therefore $$(m_1-2m_2)(m_2-2m_1)=0,$$

or $$5m_1m_2-2(m_1^2+m_2^2)=0,$$

or $$9m_1m_2-2(m_1+m_2)^2=0,$$

or, by (III), $$\frac{9a}{b}-\frac{8h^2}{b^2}=0,$$

or $$8h^2=9ab.$$

Since the steps are reversible, the condition is sufficient as well as necessary.

Ex. 3. Prove that the condition that the gradient of one of the lines represented by $ax^2+2hxy+by^2=0$ should be the square of the gradient of the other is that $ab(a+b)=6abh-8h^3$. Is the condition sufficient?

Ex. 4. Find the necessary and sufficient condition that of the lines represented by $ax^2+2hxy+by^2=0$, twice the gradient of one with thrice the gradient of the other should equal 5.

Ex. 5. Find the necessary and sufficient condition that one of the lines $ax^2+2hxy+by^2=0$ should coincide with one of the lines
$$a'x^2+2h'xy+b'y^2=0.$$

Ex. 6. Prove that the equation
$$6x^2+5xy-4y^2=0$$
represents the parallels through the origin to the lines
$$6x^2+5xy-4y^2-7x+9y-5=0.$$

Ex. 7. Find the equations of the parallels through the origin to the lines $$6x^2+5xy-6y^2-7x-4y+2=0.$$

8. Prove that the pair of lines

$$bx^2-2hxy+ay^2=0$$

e perpendiculars through the origin to the pair of lines

$$ax^2+2hxy+by^2=0.$$

9. Find the equations of (i) the parallels, (ii) the perpendiculars gh the origin to the lines represented by

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

. **Angle between the Lines $ax^2+2hxy+by^2=0$.** Let the lines represented by

$$ax^2+2hxy+by^2=0$$

$$y=m_1x \;......(1) \quad \text{and} \quad y=m_2x \;......(2);$$

$$m_1+m_2=-\frac{2h}{b} \;......(3) \quad \text{and} \quad m_1m_2=\frac{a}{b} \;......(4),$$

quation (III) of § 41.

et θ be the angle between lines (1) and (2); then, by § 35,

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}. \;......................(5)$$

ut $\quad (m_1-m_2)^2=(m_1+m_2)^2-4m_1m_2$

$$=\frac{4h^2}{b^2}-\frac{4a}{b}=\frac{4(h^2-ab)}{b^2}, \text{ from (3) and (4).}$$

herefore $\quad m_1-m_2=\pm\frac{2\sqrt{h^2-ab}}{b}.$

lso, from (4), $\quad 1+m_1m_2=1+\frac{a}{b}=\frac{a+b}{b}.$

Ience, $\quad \frac{m_1-m_2}{1+m_1m_2}=\pm\frac{2\sqrt{h^2-ab}}{a+b},$

therefore, by (5), $\quad \tan\theta=\frac{2\sqrt{h^2-ab}}{a+b},$

he ambiguity in sign be dropped.

Cor. The lines are perpendicular if $a+b=0$.

x. 1. Find the tangent of the acute angle between the lines n by the equation $3x^2-10xy+3y^2=0$. Find the angle from a k of Tables.

Ex. 2. Prove that the angle between the pair of lines

$$3x^2-7xy+4y^2=0$$

is equal to the angle between the pair $6x^2-5xy+y^2=0$.

Ex. 3. Prove that the angle between the pair of lines specified by the equation $2x^2-5xy-3y^2+x+11y-6=0$ is equal to the angle between the pair specified by $2x^2-5xy-3y^2=0$.

Ex. 4. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines inclined at an angle θ, prove that

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}.$$

Ex. 5. If $6x^2-11xy-10y^2-19y+c=0$ represents two straight lines, find the equations of the lines and the tangent of the angle between them.

43. The Bisectors of the Angles between the Lines

$$\mathbf{ax^2+2hxy+by^2=0.}$$

Let the lines represented by the equation

$$ax^2+2hxy+by^2=0$$

be $\qquad y-m_1x=0 \quad \text{and} \quad y-m_2x=0,$

so that $m_1+m_2=-2h/b$ and $m_1m_2=a/b$, by (III) of § 41.

Let $\qquad m_1=\tan\theta_1 \quad \text{and} \quad m_2=\tan\theta_2.$

Then if $\tan\theta$ is the gradient of a bisector, we may write

$$\tan 2\theta=\tan(\theta_1+\theta_2);$$

therefore
$$\frac{2\tan\theta}{1-\tan^2\theta}=\frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2}$$

$$=\frac{m_1+m_2}{1-m_1m_2}=\frac{-2h/b}{1-a/b}=\frac{2h}{a-b}.$$

Hence $\qquad (a-b)\tan\theta=h(1-\tan^2\theta)$

or $$h\tan^2\theta+(a-b)\tan\theta-h=0. \quad \ldots\ldots\ldots (1)$$

If the equations of the bisectors are

$$y-n_1x=0 \quad \text{and} \quad y-n_2x=0, \quad \ldots\ldots\ldots (2)$$

n_1 and n_2 are the roots of (1), so that

$$n_1+n_2=-\frac{a-b}{h} \quad \text{and} \quad n_1n_2=-1. \quad \ldots\ldots (3)$$

But, from (2), the equation of the bisectors is

$$(y-n_1x)(y-n_2x)=0$$

or
$$y^2-(n_1+n_2)xy+n_1n_2x^2=0.$$

Hence, from (3), the equation of the bisectors becomes

$$y^2+\frac{a-b}{h}xy-x^2=0,$$

or
$$hx^2-(a-b)xy-hy^2=0,$$

or
$$\frac{x^2-y^2}{a-b}=\frac{xy}{h}.$$

Ex. 1. If (x, y) is a point on a bisector of the angles between $y-m_1x=0$ and $y-m_2x=0$, show that

$$\frac{(y-m_1x)^2}{1+m_1^2}=\frac{(y-m_2x)^2}{1+m_2^2},$$

and deduce the equation of the bisectors of the angles between the line-pair $ax^2+2hxy+by^2=0$.

Ex 2. If m is the gradient of a bisector of the angles formed by the lines $y-m_1x=0$ and $y-m_2x=0$, show that

$$\frac{m-m_1}{1+mm_1}=\frac{m_2-m}{1+mm_2},$$

and deduce the equation of the bisectors of the angles between the line-pair $ax^2+2hxy+by^2=0$.

44. Harmonic Ranges. If A, B, C, D are four points on an axis such that

$$\frac{AC}{CB}=-\frac{AD}{DB}, \quad\text{.........................(1)}$$

that is, such that AB is divided internally and externally in the same ratio at C and D (Fig. 32), then AB is said to

FIG. 32.

be divided harmonically at C and D. Equation (1) may also be written in the form

$$\frac{CA}{AD}=-\frac{CB}{BD},$$

so that CD is at the same time divided harmonically by A and B. A, B, C, D form a **harmonic range**; A and B form one pair of **conjugate points** of the range, C and D form another pair. We also say that $(ABCD)$ is a harmonic range.

If A, B, C, D lie on an x-axis, origin O, and if their abscissae are x_1, x_2, x_3, x_4 respectively, then

$$AC = OC - OA = x_3 - x_1; \quad CB = x_2 - x_3;$$
$$AD = x_4 - x_1; \qquad DB = x_2 - x_4.$$

Hence $$\frac{x_3 - x_1}{x_2 - x_3} = -\frac{x_4 - x_1}{x_2 - x_4},$$

which may be written in the form

$$(x_1 + x_2)(x_3 + x_4) = 2(x_1x_2 + x_3x_4). \quad \dots\dots\dots\dots(2)$$

Since these steps are reversible, relations (1) and (2) are equivalent.

The relation (2) has three important forms as follows.

I. If $(ABCD)$ is a harmonic range, and O is the middle point of AB, then $$OA^2 = OC \cdot OD,$$
and conversely.

Take O to be the origin of the axis on which lie the points A, B, C, D. Then we may put $x_2 = -x_1$ or $x_1 + x_2 = 0$ in (2), when we obtain, after division by 2,

$$0 = -x_1^2 + x_3x_4$$

or $$x_1^2 = x_3x_4.$$

But $$x_1^2 = OA^2, \quad x_3 = OC, \quad x_4 = OD.$$

Hence $$OA^2 = OC \cdot OD.$$

Since the steps are reversible, the converse holds.

II. If $(ABCD)$ is a harmonic range, then

$$\frac{2}{AB} = \frac{1}{AC} + \frac{1}{AD},$$

and conversely.

Take A to be the origin of the axis on which lie the points. Then we may put $x_1 = 0$ in equation (2), when we obtain

$$x_2(x_3 + x_4) = 2x_3x_4.$$

Divide both sides by $x_2x_3x_4$;

then $$\frac{1}{x_4}+\frac{1}{x_3}=\frac{2}{x_2}.$$

But $$x_2=AB, \quad x_3=AC, \quad x_4=AD.$$

Therefore $$\frac{2}{AB}=\frac{1}{AC}+\frac{1}{AD}.$$

Since the steps are reversible, the converse holds.

III. If the roots of the equations

$$ax^2+2bx+c=0 \quad \text{and} \quad a'x^2+2b'x+c'=0$$

are the abscissae of A and B, and C and D, where $(ABCD)$ is a harmonic range,

then $$ac'+a'c=2bb',$$

and conversely.

For $$x_1+x_2=-\frac{2b}{a}, \quad x_1x_2=\frac{c}{a};$$

$$x_3+x_4=-\frac{2b'}{a'}, \quad x_3x_4=\frac{c'}{a'}.$$

Substitute in (2);

then $$\frac{4bb'}{aa'}=2\left(\frac{c}{a}+\frac{c'}{a'}\right)$$

or $$ac'+a'c=2bb'.$$

Since the steps are reversible, the converse holds.

Ex. 1. If $(ABCD)$ is a harmonic range and O' bisects CD, prove that $BC.BD=BA.BO'$, and conversely.

Ex. 2. If $(ABCD)$ is a harmonic range, O and O' the middle points of AB and CD, prove that

(i) $AC.AO=AD.OC$; (ii) $AB^2+CD^2=4OO'^2$;

(iii) $AB.CD+2AD.BC=0$; (iv) $CA.CB+DA.DB=CD^2$.

Ex. 3. If $(ABCD)$ is a harmonic range and P any point on the line of the range, prove that

(i) $PA.BC+PB.AD+PC.DB+PD.CA=0$;

(ii) $2\dfrac{PB}{AB}=\dfrac{PC}{AC}+\dfrac{PD}{AD}$.

45. Fundamental Theorem. The following theorem is of fundamental importance.

Let $(ABCD)$ be a harmonic range, and let lines be drawn from any point O outside the line of the range to pass through A, B, C, D (Fig. 33). If any line $A'B'C'D'$ be drawn to cut OA, OB, OC, OD in A', B', C', D' respectively, then $(A'B'C'D')$ is also a harmonic range.

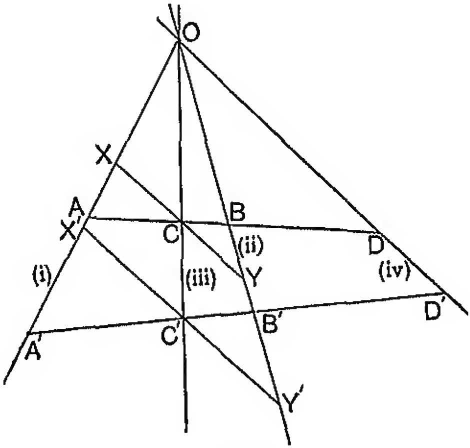


FIG. 33.

Through C and C' draw XCY and $X'C'Y'$ parallel to OD to meet OA and OB in X and Y, and X' and Y' respectively.

Then $$\frac{AC}{CB} = -\frac{AD}{DB},$$

since $(ABCD)$ is a harmonic range;

therefore $$\frac{AC}{AD} = -\frac{CB}{DB}. \qquad \text{.......................(1)}$$

But $$\frac{AC}{AD} = \frac{XC}{OD},$$

since triangles ACX, ADO are similar,

and $$\frac{CB}{DB}=\frac{YC}{OD},$$

since triangles CBY, DBO are similar.

Substituting these values in (1), we obtain

$$\frac{XC}{OD}=-\frac{YC}{OD};$$

therefore $$XC=-YC.$$

Hence C is the middle point of XY. ....(2)

But $\dfrac{CX}{C'X'}=\dfrac{OC}{OC'}$ and $\dfrac{YC}{Y'C'}=\dfrac{OC}{OC'}$, from similar triangles.

Therefore $$\frac{CX}{C'X'}=\frac{YC}{Y'C'};$$

hence $C'X'=Y'C'$, since $CX=YC$, by (2).

We may now reverse the steps from (2) to (1), using dashed letters, whence we obtain

$$\frac{A'C'}{C'B'}=-\frac{A'D'}{D'B'};$$

or $(A'B'C'D')$ is a harmonic range.

46. Harmonic Pencils. If four straight lines OA, OB, OC, OD are drawn from a point O to four points A, B, C, D, which are such that $(ABCD)$ is a harmonic range, the four lines, called *rays*, form a *harmonic pencil*; and if any line, called a *transversal*, be drawn to meet the rays, the four points of intersection with the rays form a harmonic range, by the theorem of § 45. $O(ABCD)$ is called a harmonic pencil; OA and OB form one pair of *conjugate rays*, OC and OD the other pair. Certain forms of the **equations of the rays of a harmonic pencil** are important; these we proceed to investigate.

I. The lines $y=\pm kx$ are harmonically conjugate with respect to the lines $x=0$ and $y=0$ for any value of k.

Consider the four lines

$$y=kx \text{ (i)},\quad y=-kx \text{ (ii)},\quad x=0 \text{ (iii)},\quad y=0 \text{ (iv)}.$$

On (i) take a point $A(x_1, y_1)$; through A draw a line to meet (iii) in C and (iv) in D. Mark on the line the point $B(x_2, y_2)$, the harmonic conjugate of A with respect to C and D.

Then $$\frac{AC}{CB} = -\frac{AD}{DB} = \frac{m}{n}, \text{ say,}$$

so that C is the point

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

and D is the point

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right).$$

It remains to show that B lies on (ii).

C lies on (iii), therefore $\dfrac{mx_2 + nx_1}{m+n} = 0$ or $\dfrac{m}{n} = -\dfrac{x_1}{x_2}$. ....(v)

D lies on (iv), therefore $\dfrac{my_2 - ny_1}{m-n} = 0$ or $\dfrac{m}{n} = \dfrac{y_1}{y_2}$. ......(vi)

Hence, from (v) and (vi),

$$\frac{x_1}{x_2} + \frac{y_1}{y_2} = 0 \quad \text{or} \quad \frac{y_1}{x_1} + \frac{y_2}{x_2} = 0.$$

But $\dfrac{y_1}{x_1} = k$, since A lies on (i); therefore

$$k + \frac{y_2}{x_2} = 0 \quad \text{or} \quad y_2 = -kx_2,$$

that is, B lies on (ii).

The proof does not assume that the axes are rectangular, so that we have at the same time an analytical proof of the Fundamental Theorem. If the axes are rectangular, there is an easy geometrical proof depending on Euc. VI. 3, which is left to the reader.

II. The lines $ax + by + c = \pm k(a'x + b'y + c')$ are harmonically conjugate with respect to the lines

$$ax + by + c = 0 \quad \text{and} \quad a'x + b'y + c' = 0,$$

for any value of k.

Consider the four lines

$$ax+by+c=k(a'x+b'y+c'), \qquad \text{(i)}$$

$$ax+by+c=-k(a'x+b'y+c'), \qquad \text{(ii)}$$

$$ax+by+c=0, \quad \text{(iii)} \qquad a'x+b'y+c'=0. \quad \text{(iv)}$$

On (i) take a point $A(x_1, y_1)$ (Fig. 33, p. 90); through A draw a line to meet (iii) in C and (iv) in D. Mark on the line the point $B(x_2, y_2)$, which is the harmonic conjugate of A with respect to C and D.

Then $$\frac{AC}{CB}=-\frac{AD}{DB}=\frac{m}{n}, \text{ say},$$

and therefore the coordinates of C and D have the same form as in Case I.

It remains to show that B lies on (ii).

C lies on (iii); therefore

$$\frac{a(mx_2+nx_1)}{m+n}+\frac{b(my_2+ny_1)}{m+n}+c=0,$$

that is, $$\frac{m}{n}=-\frac{ax_1+by_1+c}{ax_2+by_2+c}. \qquad \text{(v)}$$

D lies on (iv); therefore

$$\frac{a'(mx_2-nx_1)}{m-n}+\frac{b'(my_2-ny_1)}{m-n}+c'=0,$$

that is, $$\frac{m}{n}=\frac{a'x_1+b'y_1+c'}{a'x_2+b'y_2+c'}. \qquad \text{(vi)}$$

Hence, from (v) and (vi),

$$\frac{ax_1+by_1+c}{ax_2+by_2+c}+\frac{a'x_1+b'y_1+c'}{a'x_2+b'y_2+c'}=0$$

or $$\frac{ax_1+by_1+c}{a'x_1+b'y_1+c'}+\frac{ax_2+by_2+c}{a'x_2+b'y_2+c'}=0.$$

But $\dfrac{ax_1+by_1+c}{a'x_1+b'y_1+c'}=k$, since A lies on (i); therefore

$$k+\frac{ax_2+by_2+c}{a'x_2+b'y_2+c'}=0 \quad \text{or} \quad ax_2+by_2+c=-k(a'x_2+b'y_2+c'),$$

that is, B lies on (ii).

Note that this is also an analytical proof of the Fundamental Theorem.

III. The lines

$$ax^2+2hxy+by^2=0,\ \text{(i) and (ii)},$$

are harmonically conjugate with respect to the lines

$$a'x^2+2h'xy+b'y^2=0,\ \text{(iii) and (iv)},$$

if
$$ab'+a'b=2hh'.$$

Let the transversal $y=1$ meet the lines (i) and (ii) in points denoted by their abscissae x_1 and x_2, and the lines (iii) and (iv) in x_3 and x_4 (Fig. 34). Then (i), (ii), (iii), (iv)

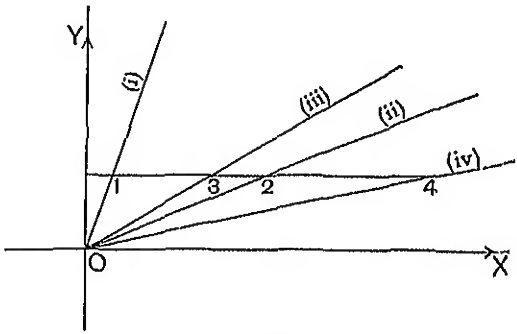


FIG. 34.

form a harmonic pencil if $(x_1x_2x_3x_4)$ is a harmonic range. But x_1 and x_2 are the roots of $ax^2+2hx+b=0$; and x_3 and x_4 are the roots $a'x^2+2h'x+b'=0$. Therefore $(x_1x_2x_3x_4)$ is a harmonic range if $ab'+a'b=2hh'$, by § 44, III.

The method of proof used in I and II may be used here also, and is left as an exercise to the reader.

EXERCISES XIII.

1. If the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ meet the line

$$ax+by+c=0$$

in C, prove that
$$\frac{AC}{CB}=-\frac{ax_1+by_1+c}{ax_2+by_2+c}.$$

2. If a transversal DEF meet the sides BC, CA, AB of triangle ABC in D, E, F respectively, prove that

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=-1,$$

and conversely (**Menelaus's Theorem**).

3. If the line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ be cut in C by the line joining (x_3, y_3) and (x_4, y_4), prove that

$$\frac{AC}{CB}=-\frac{x_1y_3-x_3y_1+x_3y_4-x_4y_3+x_4y_1-x_1y_4}{x_2y_3-x_3y_2+x_3y_4-x_4y_3+x_4y_2-x_2y_4}.$$

4. If the lines joining the vertices A, B, C of triangle ABC to any point S meet the opposite sides BC, CA, AB respectively in D, E, F, prove that

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=+1,$$

and conversely (**Ceva's Theorem**).

5. $(ABCD)$ is a harmonic range on the x-axis. The abscissae of A and B are the roots of the equation $x^2-7x+5=0$ and the abscissa of C is -1; find the abscissa of D.

6. The points on the x-axis denoted by $x^2+3x-2=0$ are harmonically conjugate with respect to the pair denoted by $x^2+5x+q=0$; find the value of q.

7. The points P, Q are harmonic conjugates with respect to the points $x_1=4$ and $x_2=7$, and also with respect to $x_3=-5$, $x_4=-2$, where all the points lie on the x-axis; find the abscissae of P and Q.

8. The three pairs of points $x=2$ and $x=6$; $x=3$ and $x=4$; $x=-1$ and $x=k$ have a common segment of harmonic section; find the value of k.

9. Prove that the pair of points denoted by

$$(ax+b)(b'x+c')=(a'x+b')(bx+c)$$

is harmonically conjugate with respect to both pairs of points denoted by $ax^2+2bx+c=0$ and $a'x^2+2b'x+c'=0$.

10. If the points x_1, x_2, x_3, x_4 on the x-axis form a harmonic range, so do the points y_1, y_2, y_3, y_4 on the y-axis, where

$$y=\frac{ax+b}{cx+d}.$$

11. Prove that the lines

$$2x^2-2xy-y^2=0 \quad\text{and}\quad x^2+3xy-y^2=0$$

form a harmonic pencil.

12. The lines

$$5x^2-xy-y^2=0 \quad\text{and}\quad 4x^2+kxy-3y^2=0$$

form a harmonic pencil; find k.

13. The equations of three rays of a harmonic pencil are

$$2x-y+3=0,\quad 3x-4y+7=0,\quad x-y+2=0;$$

find the equation of the fourth ray, the first two rays being conjugate rays.

14. Prove that the four lines

$$5x-2y+1=0,\qquad x+2y-4=0,$$
$$13x-10y+11=0,\quad 17x-2y-5=0$$

form a harmonic pencil.

15. The lines $y=m_1x$ and $y=m_2x$ are harmonically conjugate with respect to $y=m_3x$ and $y=m_4x$, if

$$\frac{m_1-m_3}{m_3-m_2}\cdot\frac{m_4-m_2}{m_1-m_4}=-1.$$

16. The lines $y=m_1x$ and $y=m_2x$ are harmonically conjugate with respect to the lines $ax^2+2hxy+by^2=0$, if

$$a+h(m_1+m_2)+bm_1m_2=0.$$

17. If $y=m_1x$, $y=m_2x$, $y=m_3x$, $y=m_4x$ form a harmonic pencil, so do

$$y=n_1x,\quad y=n_2x,\quad y=n_3x,\quad y=n_4x,$$

where $$n=\frac{a+bm}{c+dm}.$$

18. If one pair of conjugate rays of a harmonic pencil are at right angles, they are the bisectors of the angles formed by the other pair, and conversely.

47. Three or More Lines through the Origin.

Let $\quad y-m_1x=0$ (1), $\quad y-m_2x=0$ (2), $\quad y-m_3x=0$ (3)

be three lines through the origin. Then the equation

$$(y-m_1x)(y-m_2x)(y-m_3x)=0 \quad\text{..............}(4)$$

represents the lines (1), (2), (3).

If (4) be expanded, it takes the form

$$y^3-(m_1+m_2+m_3)y^2x+(m_2m_3+m_3m_1+m_1m_2)yx^2 -m_1m_2m_3x^3=0.$$

This is of the form

$$ax^3+bx^2y+cxy^2+dy^3=0, \quad\text{.................}(5)$$

so that (5) represents three straight lines through the origin of gradients m_1, m_2, m_3, where

$$m_1+m_2+m_3=-\frac{c}{d};\quad m_2m_3+m_3m_1+m_1m_2=\frac{b}{d};$$
$$m_1m_2m_3=-\frac{a}{d}. \qquad\qquad (6)$$

Similarly, the equation

$$ax^4+bx^3y+cx^2y^2+dxy^3+ey^4=0$$

represents four straight lines through the origin; and so on.

Ex. 1. Find the condition that two of the three lines

$$ax^3+bx^2y+cxy^2+dy^3=0$$

should be at right angles.

Let the gradients of the lines be m_1, m_2, m_3.

Then either $\quad (1+m_1m_2)=0 \quad$ or $\quad (1+m_2m_3)=0 \quad$ or $\quad (1+m_3m_1)=0.$

Therefore $\quad (1+m_1m_2)(1+m_2m_3)(1+m_3m_1)=0,$

that is, $1+(m_1m_2+m_2m_3+m_3m_1)+m_1m_2m_3(m_1+m_2+m_3)+m_1^2m_2^2m_3^2=0;$

therefore, by (6) above,

$$1+\frac{b}{d}+\frac{a}{d}\cdot\frac{c}{d}+\frac{a^2}{d^2}=0$$

or
$$a^2+ac+bd+d^2=0.$$

Since the steps are reversible, the condition is sufficient as well as necessary.

Ex. 2. Find the necessary and sufficient condition that the gradient of one of the lines specified by the equation

$$y^3-p_1y^2x+p_2yx^2-p_3x^3=0$$

be equal to the product of the other two.

Ex. 3. If the gradient of one of the three lines

$$y^3-p_1y^2x+p_2yx^2-p_3x^3=0$$

be equal to the sum of the other two, prove that

$$p_1^3-4p_1p_2+8p_3=0;$$

and show that the condition is sufficient.

Ex. 4. If the gradients of the four lines specified by the equation

$$y^4-p_1y^3x+p_2y^2x^2-p_3yx^3+p_4x^4=0$$

be in proportion, prove that $p_3^2=p_4p_1^2$; and prove that this condition is sufficient.

48. Change of Origin. Let $X'OX$, $Y'OY$ be a rectangular system of reference, and let $\xi'\omega\xi$, $\eta'\omega\eta$ be another rectangular system of reference, the x- and ξ-axes being parallel and the y- and η-axes also parallel (Fig. 35).

Let (x, y) be the coordinates of a point P referred to $X'OX$, $Y'OY$; let (ξ, η) be the coordinates of P referred to $\xi'\omega\xi$, $\eta'\omega\eta$; let (h, k) be the coordinates of ω referred to $X'OX$, $Y'OY$;

then $$\mathbf{x} = \xi + \mathbf{h} \quad \text{and} \quad \mathbf{y} = \eta + \mathbf{k}.$$

Proof. Let $\eta'\omega\eta$ meet $X'OX$ in H, and let MP, NP be the ordinates of P referred to the two systems, M lying on $X'OX$ and N on $\xi'\omega\xi$.

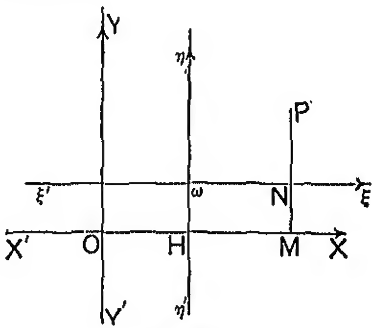


FIG. 35.

Then $$x = OM = OH + HM = OH + \omega N = h + \xi = \xi + h;$$
$$y = MP = MN + NP = H\omega + NP = k + \eta = \eta + k.$$

For example, if the bisector of the angle XOY be referred to parallel axes through (2, 1), its equation referred to the new axes $\xi'\omega\xi$, $\eta'\omega\eta$ is
$$\eta = \xi + 1.$$

For $y = x$ (i) is its equation referred to the first axes; put
$$x = \xi + h = \xi + 2 \quad \text{and} \quad y = \eta + k = \eta + 1$$
in (i); then we obtain
$$\eta + 1 = \xi + 2 \quad \text{or} \quad \eta = \xi + 1$$
as the equation of the line referred to the new axes. This is easily verified from a figure.

49. Rotation of Axes. It is sometimes useful to change from one system of rectangular axes of reference to another formed by rotating the old axes through an angle; we proceed to find the formulae necessary for the transformation of equations.

Let $\xi' O \xi$, $\eta' O \eta$ be rectangular axes obtained by rotating the rectangular axes $X'OX$, $Y'OY$ through an angle θ (Fig. 36); let (x, y) and (ξ, η) be the coordinates of a point P referred to the two systems.

Then
$$x = \xi \cos\theta - \eta \sin\theta,$$
$$y = \xi \sin\theta + \eta \cos\theta.$$

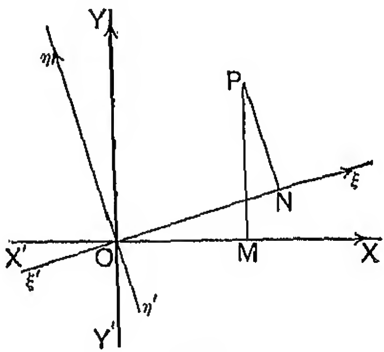


FIG. 36.

Let angle $\xi OP = \phi$.

Then
$$\begin{aligned} x &= OP \cos(\theta + \phi) \\ &= OP(\cos\theta \cos\phi - \sin\theta \sin\phi) \\ &= (OP \cos\phi) \cdot \cos\theta - (OP \sin\phi) \cdot \sin\theta \\ &= \xi \cos\theta - \eta \sin\theta. \end{aligned}$$

Also
$$\begin{aligned} y &= OP \sin(\theta + \phi) \\ &= OP(\sin\theta \cos\phi + \cos\theta \sin\phi) \\ &= (OP \cos\phi) \cdot \sin\theta + (OP \sin\phi) \cdot \cos\theta \\ &= \xi \sin\theta + \eta \cos\theta. \end{aligned}$$

Ex. 1. What is the equation of the line $2x-3y=5$ (i) referred to parallel axes $\xi'\xi$ and $\eta'\eta$ drawn through the point $(-1, -3)$ referred to the x- and y-axes?

Put $x=\xi+h=\xi-1$ and $y=\eta+k=\eta-3$ in equation (i). We obtain $2(\xi-1)-3(\eta-3)=5$ or $2\xi-3\eta+2=0$ as the equation of line (i), when referred to the ξ- and η-axes.

Ex. 2. Find the equation of the line $3x-2y=5$ referred to parallel axes $\xi'\xi$ and $\eta'\eta$ through the point $(1, -1)$ referred to the old axes.

Proceeding as in Ex. 1, we obtain as the equation

$$3(\xi+1)-2(\eta-1)=5 \quad \text{or} \quad 3\xi=2\eta. \quad \text{.................(ii)}$$

It appears that the line passes through the *new origin*, as it should, since $(1, -1)$ lies on $3x-2y=5$. Since the gradient in each case is $3/2$, it is clear that equation (ii) is correct.

EXERCISES XIV.

1. The two lines $3x-4y+2=0$ and $x-y+1=0$, when referred to parallel axes $\xi'\xi$ and $\eta'\eta$, are represented by the equations $3\xi=4\eta$ and $\xi=\eta$ respectively. Find the coordinates of the new origin referred to the x- and y-axes.

2. The lines $(x-y+2)(x+y+4)=0$, when referred to parallel axes through the point (h, k), are represented by the equation $\xi^2-\eta^2=0.$

Find h and k.

3. Prove that the parallels through the origin to the lines

$$2x^2+5xy-3y^2-3x+5y-2=0$$

are the lines $2x^2+5xy-3y^2=0.$

4. Find the equation of the parallels through the origin to the lines

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

5. If θ is the angle between the lines

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

prove that $\tan\theta=2\sqrt{h^2-ab}/(a+b)$.

Prove also that the lines are perpendicular if $a+b=0$.

6. Prove that the equation

$$a(x-p)^2+2h(x-p)(y-q)+b(y-q)^2=0$$

represents two straight lines passing through the point (p, q).

7. Prove that any pair of perpendicular lines through the point (p, q) can be represented by the equation

$$(x-p)^2+2h(x-p)(y-q)-(y-q)^2=0,$$

where h is a varying constant (parameter).

If such a pair intersect the x-axis in A and A', and the y-axis in B and B', prove that

$$\text{(i)}\ OA\,.\,OA' - p(OA + OA') + p^2 + q^2 = 0,$$
$$\text{(ii)}\ OA\,.\,OA'/OB\,.\,OB' = -1.$$

8. If the line-pair

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

be referred to parallel axes through (p, q) so that their equation takes the form
$$a\xi^2 + 2h\xi\eta + b\eta^2 = 0,$$
find p and q in terms of the coefficients of (i).

9. Parallels to the lines

$$ax^2 + 2hxy + by^2 = 0$$

are drawn through the point (p, q); find the equation of the bisectors of the angles formed by the parallels. If perpendiculars are drawn instead of parallels, find the equation of the bisectors of the angles so formed.

10. If a given line be referred to parallel axes through any point on a line parallel to the given line, then the new equation of the line is of the same form whatever be the position of the origin on the parallel line.

11. Find the equation, referred back to the x- and y-axes, of a straight line whose equation referred to parallel axes $\xi'\xi$ and $\eta'\eta$ through the point $x = 3$, $y = -2$ is $12\xi - 7\eta = 11$.

12. Through the point $x = 1$, $y = -1$ are drawn ξ- and η-axes parallel to the x- and y-axes. The equation of a line-pair referred to the ξ- and η-axes is

$$4\xi^2 + 4\xi\eta - 3\eta^2 - 2\xi + 5\eta - 2 = 0.$$

What is the equation of the line-pair referred to the x- and y-axes?

13. Find what the equation $2x - y + 3 = 0$ becomes when the axes are turned through 45°.

14. Find what the equation

$$x^2 - y^2 - 2\sqrt{2}x - 10\sqrt{2}y + 2 = 0$$

becomes if the axes are turned through 45°.

15. Transform the equation

$$(x^2 - y^2)^2 = a^2(x^2 + y^2)$$

to an equation referred to axes which bisect the angles between the original axes.

16. Transform the equation $x^3 + y^3 = a^3$ to another set of rectangular axes which have revolved in a negative direction through an angle $\frac{\pi}{4}$ from the given axes.

17. Transform the equation

$$(a^2+b^2)(x^2+y^2)+2(a^2-b^2)xy=2a^2b^2$$

to axes bisecting the angles formed by the original axes, and reduce it to its simplest form.

18. Transform the equation $x^2-2xy\cot 2\alpha-y^2=a^2$ by turning the axes through an angle $\left(\alpha-\frac{\pi}{2}\right)$, and thence graph the equation.

19. Transform the equation

$$x^2-y^2-4\sqrt{2}x-8\sqrt{2}y+4=0$$

by turning the axes through 45° and then moving the origin to the point $(-2, -6)$ referred to the axes so turned. Show that the equation then becomes $\xi\eta=14$, and thence graph the given equation.

20. If the expression

$$ax^2+2hxy+by^2+2gx+2fy+c$$

be transformed, by turning the axes (§ 40) through an angle θ, into the expression

$$a'\xi^2+2h'\xi\eta+b'\eta^2+2g'\xi+2f'\eta+c',$$

prove that
$$a'=a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta,$$
$$b'=a\sin^2\theta-2h\sin\theta\cos\theta+b\cos^2\theta,$$
$$h'=(b-a)\sin\theta\cos\theta+h(\cos^2\theta-\sin^2\theta),$$
and then show that $a+b=a'+b',\ ab-h^2=a'b'-h'^2,$
whatever be the angle θ.

21. If $7x^2+4xy+4y^2$ becomes $a'\xi^2+b'\eta^2$ by rotation of the axes, show that $a'=8,\ b'=3$ or $a'=3,\ b'=8$.

If $2x^2+12xy-3y^2$ becomes $a'\xi^2+b'\eta^2$, show that $a'=6,\ b'=-7$ or $a'=-7,\ b'=6$.

22. If the axes are turned through 45°, show that the equation

$$x^3+y^3=3axy$$

becomes
$$3\eta^2(\sqrt{2}\,.\,a+2\xi)=3\sqrt{2}\,.\,a\xi^2-2\xi^3.$$

CHAPTER VII.

THE CIRCLE.

50. Equation of a Circle. A circle is specified when we know the position of the centre and the length of the radius.

Let axes $X'OX$, $Y'OY$ be drawn, and scale-units fixed.

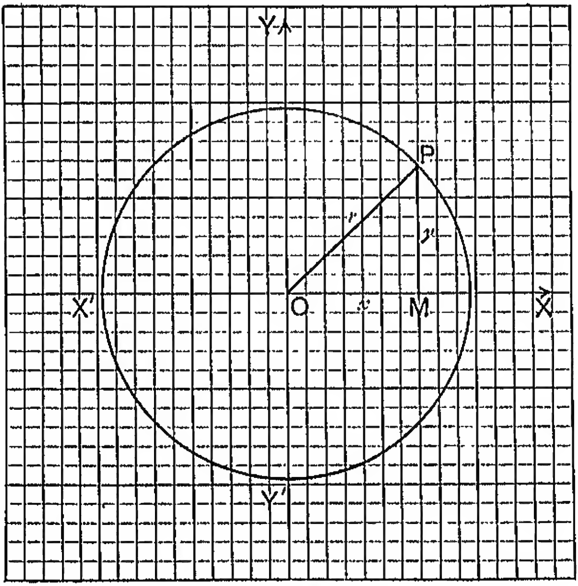


FIG. 37.

I. Suppose the origin O is the centre of the circle.
Let r = radius of circle.

Let $P(h, k)$ be a point on the circle (Fig. 37), M the projection of P on $X'OX$.

Then
$$OM^2 + MP^2 = OP^2,$$
that is,
$$h^2 + k^2 = r^2.$$

Writing x for h and y for k to indicate a variable point on the circle, we get
$$\mathbf{x^2 + y^2 = r^2}$$
as the equation of the circle.

II. Suppose O, the origin, is not the centre of the circle. Then the centre of the circle, as well as the radius, must be specified.

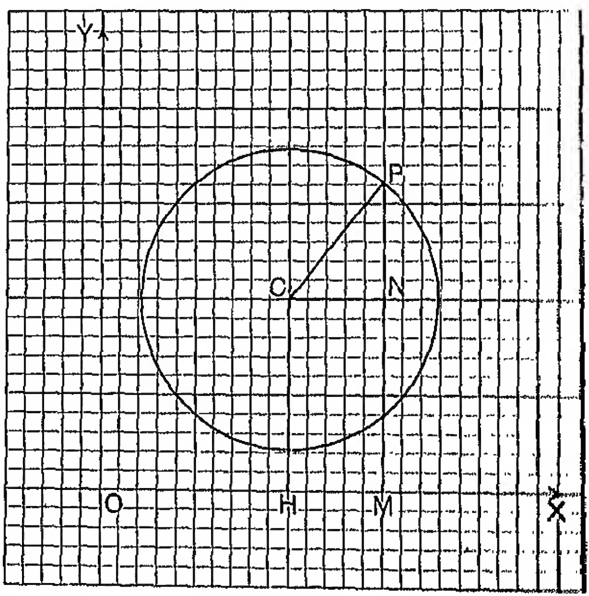


FIG. 38.

Let (a, b) be the coordinates of the centre.
Let r = radius of circle.
Let $P(h, k)$ be any point on the circle.

We have to translate the defining property of a circle into an equation connecting h and k with the constants a, b, r which specify the circle.

Let C be the centre of the circle.

Let H, M (Fig. 38) be the projections of C, P on $X'OX$.

Let the parallel to $X'OX$ through C meet MP in N.

Then $\quad OH=a,\ HC=b,\ OM=h,\ MP=k.$

Hence $\quad CN=HM=OM-OH=h-a;$ ....(1)

and $\quad NP=MP-MN=MP-HC=k-b.$ ....(2)

But the defining property of the circle gives the equation

$$CP^2=r^2.$$

Therefore $\quad CN^2+NP^2=r^2,$

or $\quad (h-a)^2+(k-b)^2=r^2$, by (1) and (2).

Writing x for h and y for k to represent a variable point on the circle, we get

$$(x-a)^2+(y-b)^2=r^2.$$

If then a system of rectangular axes be chosen, so that a circle, radius r, has its centre at the point (a, b), the circle can be represented by the equation

$$(\mathbf{x}-\mathbf{a})^2+(\mathbf{y}-\mathbf{b})^2=\mathbf{r}^2.$$

If $(x-a)^2$, $(y-b)^2$ be expanded, we shall obtain an equation containing terms in x^2, y^2, x, y and an absolute term; *but the coefficients of x^2 and y^2 will be equal, and the equation will contain no term in xy.* Hence a circle, specified with reference to rectangular axes, can be represented by equations of the three forms,

$$(\mathbf{x}-\mathbf{a})^2+(\mathbf{y}-\mathbf{b})^2=\mathbf{r}^2,$$

$$\mathbf{x}^2+\mathbf{y}^2+2\mathbf{gx}+2\mathbf{fy}+\mathbf{c}=0,$$

$$\mathbf{Ax}^2+\mathbf{Ay}^2+2\mathbf{Gx}+2\mathbf{Fy}+\mathbf{C}=0.$$

Ex. 1. Find the equations of the circles specified as follows:

(i) centre (0, 0); radius 2.

$$x^2+y^2=r^2 \text{ becomes } x^2+y^2=4.$$

(ii) centre (0, 0); radius 4. $\quad$ (iii) centre (0, 0); radius 5.

(iv) centre $(-3, 1)$; radius 2.

$(x-a)^2+(y-b)^2=r^2$ becomes $(x+3)^2+(y-1)^2=2^2$,

that is, $$x^2+y^2+6x-2y+6=0.$$

(v) centre $(-2, 2)$; radius 2. (vi) centre $(2, 1)$; radius 3.

(vii) centre $(0, 1)$; radius 1. (viii) centre $(0, -1)$; radius 2.

(ix) centre $(2, 0)$; radius 3. (x) centre $(-3, 0)$; radius 5.

(xi) centre $(2, -3)$; radius 1. (xii) centre $(-3, 4)$; radius 7.

(xiii) centre $(2, -\frac{5}{2})$; radius $\frac{1}{2}$. (xiv) centre $(-\frac{3}{2}, -\frac{1}{2})$; radius 3.

Ex. 2. Find the equation of the circle whose centre is the origin and which passes through the point $(3, 4)$.

Ex. 3. Find the equation of the circle whose centre is the point $(-5, -1)$ and which passes through the point $(-10, 11)$.

Ex. 4. Find the equation of the circle whose centre is the point $(1, -1)$ and which passes through the point $(9, 14)$.

Ex. 5. Find the points in which the circle, centre $(2, -3)$, radius 5, cuts the x-axis.

Ex. 6. Find the points in which the circle, centre $(5, 1)$, radius 13, cuts the y-axis.

Ex. 7. Find the points in which the circle, centre $(-1, \frac{1}{2})$, radius $2\frac{1}{2}$, cuts the line $y+1=0$.

Ex. 8. Find the points in which the circle, whose centre is (c, b) and which passes through $(-c, 0)$, meets the line $y=2b$.

51. The equation $x^2+y^2+2gx+2fy+c=0$ represents a circle.

The converse of the preceding section is as follows:

Any equation of the form

$$x^2+y^2+2gx+2fy+c=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

or $$Ax^2+Ay^2+2Gx+2Fy+C=0 \quad \ldots\ldots\ldots\ldots\ldots(2)$$

referred to rectangular axes, represents a circle.

For (1) and (2) are equivalent to

$$(x+g)^2+(y+f)^2=g^2+f^2-c \quad \ldots\ldots\ldots\ldots\ldots(1)'$$

and $$\left(x+\frac{G}{A}\right)^2+\left(y+\frac{F}{A}\right)^2=\left(\frac{G}{A}\right)^2+\left(\frac{F}{A}\right)^2-\frac{C}{A}, \quad \ldots\ldots\ldots(2)'$$

and therefore represent circles whose centres are respectively $(-g, -f)$ and $\left(-\frac{G}{A}, -\frac{F}{A}\right)$, and whose radii are respectively $\sqrt{g^2+f^2-c}$ and $\sqrt{G^2+F^2-AC}/A$.

Ex. 1. Prove that the equation $x^2+y^2+2x+2y+1=0$ represents a circle whose centre is the point $(-1, -1)$ and whose radius is 1.

Collecting the terms in x, and the terms in y, we get

$$(x^2+2x)+(y^2+2y)+1=0.$$

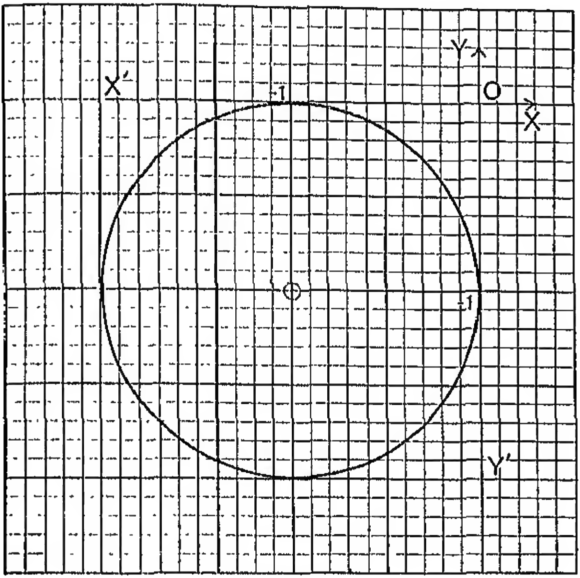


FIG. 39.

Completing the squares, we have

$$(x^2+2x+1)+(y^2+2y+1)+1=2,$$

that is, $$(x+1)^2+(y+1)^2=1^2,$$

or distance of (x, y) from $(-1, -1)$ is 1.

Hence locus of (x, y) is the circle centre $(-1, -1)$, radius 1. (See Fig. 39.)

Ex. 2. Find the centre and radius of each of the following circles:

(i) $x^2+y^2-6x-8y=0$; (ii) $x^2+y^2+6x+8y+9=0$;

(iii) $x^2+y^2-2x+2y=23$; (iv) $x^2+y^2+4x-6y=3$.

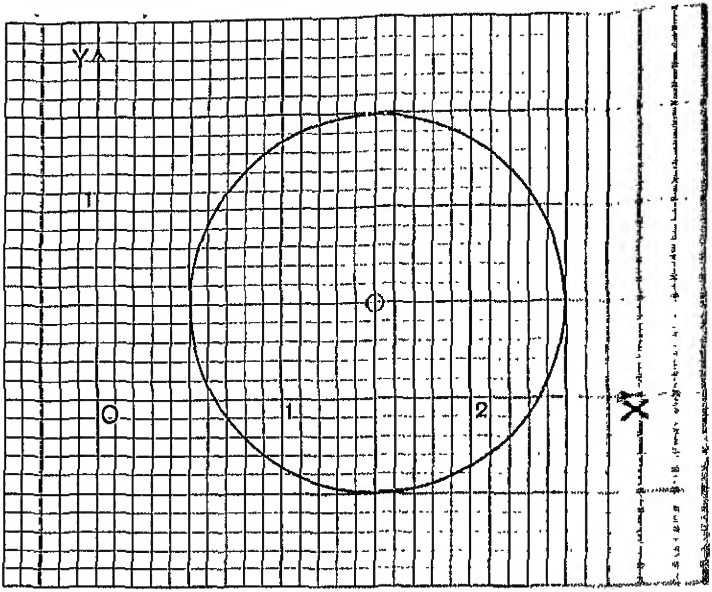


FIG. 40.

Ex. 3. Choose axes and scale-units, and draw the circles represented by the following equations. Specify the centre and radius of each:

(i) $x^2+y^2=4$; (ii) $x^2+y^2-9=0$;

(iii) $x^2+y^2-2x+4y+1=0$; (iv) $x^2+y^2-4x-6y-3=0$;

(v) $x^2+y^2+2x-2y=2$; (vi) $2x^2+2y^2-6x-2y+3=0$;

(vii) $2x^2+2y^2+10x-6y-1=0$; (viii) $3x^2+3y^2-2x+4y=0$;

(ix) $5x^2+5y^2+5x+5y=8$.

Note: (vi) may be written in the form $(x-\frac{3}{2})^2+(y-\frac{1}{2})^2=1$, which represents the circle of Fig. 40.

Ex. 4. Find the equation of the circle described on the line joining $(-1, -1)$ and $(2, 5)$ as diameter.

Let (h, k) be a point on the circle (Fig. 41).

o gradient of join of (h, k) to $(-1, -1)$ is $\dfrac{k+1}{h+1}$.

d gradient of join of (h, k) to $(2, 5)$ is $\dfrac{k-5}{h-2}$.

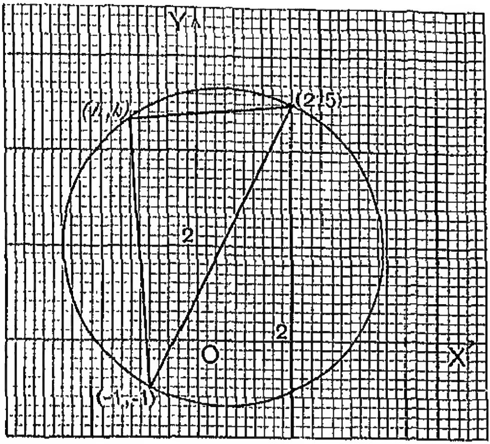


FIG. 41.

t these lines form an angle in a semicircle, or a right angle.

nce $$\frac{k+1}{h+1}\cdot\frac{k-5}{h-2}=-1 \text{ (§ 19)};$$

fore $$h^2+k^2-h-4k-7=0.$$

ite x, y for h, k to represent a variable point on the circle; equired equation is

$$x^2+y^2-x-4y=7.$$

. **Worked Examples.** We shall now work some examples he mode of translating into an analytical equation the itions that specify a circle geometrically.

. 1. A and B are the points $(2, 0)$ and $(-2, 0)$ respectively. riable point P moves so that $PA^2+2PB^2=22\frac{2}{3}$; prove that the of P is a circle of radius 2, whose centre is C, the point of tion of AB nearest to B.

; (h, k) be a point P on the locus (Fig. 42).

on $$PA^2=(h-2)^2+k^2;\quad PB^2=(h+2)^2+k^2.$$

But $$PA^2+2PB^2=22\tfrac{2}{3};$$
therefore $$(h-2)^2+k^2+2(h+2)^2+2k^2=22\tfrac{2}{3},$$
that is, $$3h^2+3k^2+4h=10\tfrac{2}{3},$$
or $$h^2+k^2+\tfrac{4}{3}h=3\tfrac{5}{9},$$
or $$(h+\tfrac{2}{3})^2+k^2=4.$$

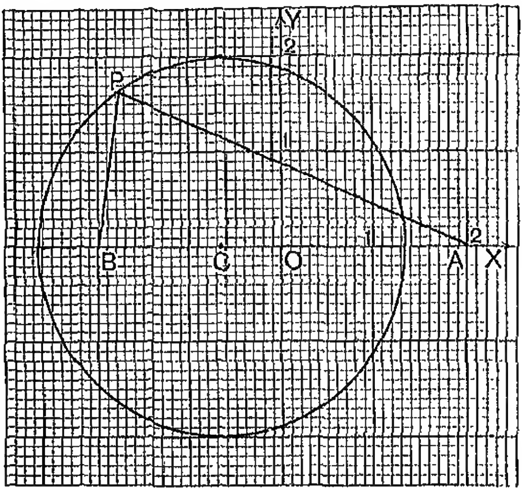


FIG. 42.

Writing x, y, for h, k to denote a variable point on the locus, we get
$$(x+\tfrac{2}{3})^2+y^2=4,$$
which is the equation of the locus, and represents a circle, radius 2 and centre $(-\frac{2}{3}, 0)$ or C.

Ex. 2. If A and B are the points $(a, 0)$ and $(b, 0)$, $b>a$, and P is a variable point above the x-axis such that angle APB is $45°$, prove that the locus of P is an arc of a circle passing through A and B.

Let $P(h, k)$ be a point on the locus.

The gradient of PA is $\dfrac{k}{h-a}$ and the gradient of PB is $\dfrac{k}{h-b}$.

Hence, using the formula $\tan\theta=\dfrac{m_1-m_2}{1+m_1m_2}$ (§ 35),

we have
$$\frac{\dfrac{k}{h-b}-\dfrac{k}{h-a}}{1+\dfrac{k}{h-b}\cdot\dfrac{k}{h-a}}=1;$$

therefore $$\frac{k(h-a)-k(h-b)}{(h-b)(h-a)+k^2}=1,$$

that is, $$h^2-h(a+b)+ab+k^2=k(b-a)$$

or $$h^2+k^2-h(a+b)+k(a-b)+ab=0.$$

Writing x, y for h, k to indicate the variable point P, we get

$$x^2+y^2-x(a+b)+y(a-b)+ab=0.$$

The locus of P is therefore a circle, passing through $(a, 0)$ and $(b, 0)$.

Ex. 3. Find the equation of the circle which passes through the three points $(1, -1)$, $(1, 4)$, $(4, -2)$.

Let the required equation be

$$x^2+y^2+2gx+2fy+c=0,$$

so that it remains to determine g, f, c.

Since $(1, -1)$ lies on the circle, we have

$$1+1+2g-2f+c=0$$

or $$2g-2f+c+2=0. \quad\text{.........................(1)}$$

Similarly, since $(1, 4)$ and $(4, -2)$ lie on the circle,

$$2g+8f+c+17=0 \quad\text{.........................(2)}$$

and $$8g-4f+c+20=0. \quad\text{.........................(3)}$$

(1), (2), (3) are three simultaneous equations in g, f, c.

From (2) subtract (1); then

$$10f+15=0 \quad\text{or}\quad f=-\tfrac{3}{2}.$$

From (3) subtract (2); then

$$6g-12f+3=0,$$

that is, $$6g+18+3=0 \quad\text{or}\quad g=-\tfrac{7}{2}.$$

Go back to (1) and substitute $g=-\frac{7}{2}$, $f=-\frac{3}{2}$; then

$$-7+3+c+2=0 \quad\text{or}\quad c=2.$$

Go back to $x^2+y^2+2gx+2fy+c=0$ and substitute $g=-\frac{7}{2}$, $f=-\frac{3}{2}$, $c=2$; then the required equation is found to be

$$x^2+y^2-7x-3y+2=0.$$

The centre of the circle is $(\frac{7}{2}, \frac{3}{2})$; the radius is $\dfrac{5\sqrt{2}}{2}$.

EXERCISES XV.

1. Find the equation of the circle which passes through the points $(2, -1)$, $(2, 3)$ and $(4, -1)$. Find also its radius and the coordinates of the centre.

2. Find the equation of the circle which passes through the origin and makes an intercept 2 on each of the axes.

3. Find the equation of the circle which passes through the origin and makes intercepts of 2, -6 on the axes of x and y respectively.

4. Find the centre of the circle which passes through the points $(2, 1)$, $(-2, 5)$, $(-3, 2)$.

5. Find the equation of the circumcircle of the triangle whose vertices are $(2, -1)$, $(5, -4)$, $(-1, -1)$. What is the radius of the circle?

6. Find the coordinates of the centres of the circles which pass through $(7, 1)$ and $(9, 5)$ and have a radius 5.

7. Trace on the same diagram the loci whose equations are

$$2x+y=3, \quad x^2+y^2=2, \quad (x-2)^2+(y-1)^2=1.$$

Find the two points common to the three loci.

8. If A and B are the fixed points $(1, 0)$, $(-1, 0)$ and P is a variable point (x, y), such that angle APB is half a right angle, prove that the equation of the locus of P is $x^2+y^2-2y=1$ or $x^2+y^2+2y=1$ according as P is above or below the x-axis. Draw the loci.

9. A and B are the fixed points $(3, 2)$, $(7, -1)$; find the equation of the circle described on AB as diameter.

10. Prove that the equation of the circle described on the join of (x_1, y_1) and (x_2, y_2) as diameter is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$$

11. If the coordinates of A, B, P are $(a, 0)$, $(b, 0)$, (x, y), where P is a variable point such that angle APB is α, prove that the two loci of P are given by the equations

$$(x-a)(x-b)+y^2 \pm (a-b)y \cot \alpha=0,$$

and assign each locus to its equation. Draw the loci.

12. A variable point P moves so that the sum of the squares of its distances from the points $(2, 0)$, $(-2, 0)$ is 16; prove that the locus of P is a circle, centre the origin, radius 2. Draw the circle.

13. A variable point P moves so that $2PA^2+3PB^2$ is 10, where A, B are the fixed points $(1, 0)$, $(-1, 0)$ respectively; prove that the locus of P is a circle whose centre is at C in AB, where $AC=\frac{4}{5}AB$. Draw the circle.

14. A variable point P moves so that PA^2-2PB^2 is 4, where A, B are the points $(1, 0)$, $(-1, 0)$ respectively; prove that the locus of P is a circle whose centre is the point obtained by producing AB its own length through B. Draw the circle.

15. A, B, C are the points $(1, 0)$, $(-1, 0)$, $(0, 3)$ respectively, and the variable point P moves so that $PA^2+PB^2+PC^2$ is 11; prove that the locus of P is a circle whose centre is the point $(0, 1)$. Draw the circle.

16. A variable point P moves so that $PA/PB=3/2$, where A and B are the points $(-5, 0)$ and $(5, 0)$ respectively; prove that the locus of P is the circle, centre $(13, 0)$, radius 12. Draw the circle.

17. A point moves so that the square of its distance from the origin is twice its ordinate; find the equation of the locus of the point and discuss the equation. Draw the locus.

18. A point P moves so that the rectangle contained by its distances from the lines $x-y=0$ and $x+y=0$ is equal to the square of its distance from the line $x=2$; find the equation of the locus of P and discuss the equation. Draw the locus.

19. Prove that the intersections of $x-2y-1=0$ and $x+y-2=0$ with $2x+y-3=0$ and $x-y-1=0$ lie on the circle $x^2+y^2-2x-y+1=0$. Draw the lines and the circle.

20. Prove that the two lines specified by the equation

$$(2x-y+3)(5x+3y-29)=0$$

intersect the two specified by

$$(x-3y+14)(x+4y+1)=0$$

on the circle whose equation is

$$9(x^2+y^2)-58x-15y-101=0.$$

21. Solve graphically the simultaneous equations

$$x^2+y^2=5;$$
$$3x+2y=4.$$

22. Solve, graphically and algebraically, the simultaneous equations

$$x^2+y^2-4x-2y+1=0;$$
$$x^2+y^2-5x+y-6=0.$$

23. Find the equation of the common chord of the circles

$$x^2+y^2+4x-4y+7=0;$$
$$x^2+y^2-6x+2y-3=0.$$

53. Equation of the Tangent to a Circle. We proceed to find the equation of the tangent at a given point on a circle specified by an equation with reference to rectangular axes.

I. Let $x^2+y^2=r^2$ specify a circle whose centre is the origin O, and whose radius is r.

Let $P(x_1, y_1)$ be a point on the circle and draw PT at right angles to OP; then PT is the tangent at P.

We have $$\text{gradient of } OP=\frac{y_1}{x_1}.$$

But PT is perpendicular to OP;

therefore $$\text{gradient of } PT=-\frac{x_1}{y_1}.$$

Hence PT is the line through (x_1, y_1) of gradient $-x_1/y_1$.

Therefore the equation of PT is

$$y-y_1=-\frac{x_1}{y_1}(x-x_1),$$

that is, $$xx_1+yy_1=x_1^2+y_1^2$$

or $$\mathbf{xx_1+yy_1=r^2}.$$

II. Let $x^2+y^2+2gx+2fy+c=0$ represent the circle whose centre is C, $(-g, -f)$.

Let $P(x_1, y_1)$ be a point on the circle and draw PT perpendicular to CP; then PT is the tangent at P.

Now $$\text{gradient of } CP=\frac{y_1+f}{x_1+g};$$

and PT is perpendicular to CP.

Therefore $$\text{gradient of } PT=-\frac{x_1+g}{y_1+f}.$$

Hence PT is the line through (x_1, y_1) of gradient $-\dfrac{x_1+g}{y_1+f}$.

Therefore the equation of PT is

$$y-y_1=-\frac{x_1+g}{y_1+f}(x-x_1),$$

that is, $$(x-x_1)(x_1+g)+(y-y_1)(y_1+f)=0$$

or $$xx_1+yy_1+gx+fy=x_1^2+y_1^2+gx_1+fy_1.$$

Add to each side gx_1+fy_1+c; then the equation of the tangent at (x_1, y_1) becomes

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c$$
$$=x_1^2+y_1^2+2gx_1+2fy_1+c$$

or
$$\mathbf{xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.}$$

It is useful to note that the equation

$$x^2+y^2+2gx+2fy+c=0$$

may be transformed into

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0,$$

by writing xx_1 for x^2, yy_1 for y^2, $(x+x_1)$ for $2x$, and $(y+y_1)$ for $2y$.

Ex. 1. Prove that $3x+4y=25$ is the tangent at $(3, 4)$ to $x^2+y^2=25$.

Ex. 2. Find the equation of the tangent

(i) at $(2, 3)$ to $x^2+y^2=13$; (ii) at $(-1, 1)$ to $x^2+y^2=2$;
(iii) at $(2, -1)$ to $x^2+y^2=5$; (iv) at $(3, 5)$ to $x^2+y^2-2x-4y=8$;
(v) at $(1, -2)$ to $x^2+y^2-4x+6y+11=0$;
(vi) at $(2, -3)$ to $x^2+y^2+4y=1$;
(vii) at $(-3, -2)$ to $x^2+y^2+10x+2y+21=0$;
(viii) at $(\frac{3}{2}, \frac{3}{2})$ to $2x^2+2y^2+2x+6y=21$;
(ix) at $(\frac{5}{3}, -\frac{1}{2})$ to $36(x^2+y^2)+24x-36y=167$.

Ex. 3. Show that the equation of the tangent at the point (x_1, y_1) on the circle whose equation is

$$(x-a)^2+(y-b)^2=r^2$$

may be put in the form

$$(x_1-a)(x-x_1)+(y_1-b)(y-y_1)=0.$$

Ex. 4. Show (i) that the point $(a+r\cos\theta,\ b+r\sin\theta)$ lies on the circle given by the equation

$$(x-a)^2+(y-b)^2=r^2,$$

whatever be the value of θ, and (ii) that the equation of the tangent at the point is $(x-a)\cos\theta+(y-b)\sin\theta=r.$

54. Equations of Secant and Tangent. We may obtain the equation of the tangent to a circle without assuming that it is the perpendicular to the radius to the point of contact, but we must in that case have some other property of the tangent on which to base our reasoning.

Suppose the secant PAB (Fig. 43, p. 121) to turn about P until it takes the position of the tangent PT; in this position the two points in which the line cuts the circle have become *coincident*. We may therefore define the tangent at T to be the straight line which meets the circle in *two coincident points* at T.

We shall find the equation of a secant in two ways, taking the circle whose equation is

$$x^2+y^2=r^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Gradient Method. The equation of the line joining (x_1, y_1) and (x_2, y_2) is

$$y-y_1=\frac{y_1-y_2}{x_1-x_2}(x-x_1). \quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Equation (2) is true whether the points lie on the circle (1) or not; we must transform equation (2) so that the points (x_1, y_1), (x_2, y_2) may be restricted to the circle. The conditions that these points lie on the circle (1) are

$$x_1^2+y_1^2=r^2 \quad\text{and}\quad x_2^2+y_2^2=r^2;\ldots\ldots\ldots\ldots(3)$$

and therefore, by subtraction,

$$x_1^2-x_2^2+y_1^2-y_2^2=0$$

or

$$(x_1-x_2)(x_1+x_2)+(y_1-y_2)(y_1+y_2)=0$$

so that the gradient of the secant is given by the equation

$$\frac{y_1-y_2}{x_1-x_2}=-\frac{x_1+x_2}{y_1+y_2}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Equation (2) now becomes

$$y-y_1=-\frac{x_1+x_2}{y_1+y_2}(x-x_1)$$

or

$$(x_1+x_2)x+(y_1+y_2)y=r^2+x_1x_2+y_1y_2, \quad\ldots\ldots\ldots(5)$$

since

$$x_1^2+y_1^2=r^2.$$

It is easy to verify that equation (5) represents the line through (x_1, y_1) and (x_2, y_2), provided these points lie on the circle (1); apart therefore from the particular process by which the equation has been reached, we know that it represents the required secant.

If we now suppose (x_2, y_2) to become coincident with (x_1, y_1), we get the equation of the tangent at (x_1, y_1),

$$2x_1x+2y_1y=r^2+x_1^2+y_1^2=2r^2,$$

that is,
$$x_1x+y_1y=r^2. \quad \text{......................(6)}$$

Burnside's Method. The following ingenious method is due to Burnside (Salmon's *Conic Sections*, § 85).

The equation of the secant through the two points (x_1, y_1) and (x_2, y_2) on the circle (1) is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=x^2+y^2-r^2. \quad \text{........(7)}$$

This equation, though apparently of the second degree in x, y, is really of the first, because the terms in x^2 and y^2 cancel; it is therefore the equation of some straight line. Next, since (x_1, y_1) lies on the circle (1), the right side of equation (7) will be zero when $x=x_1$ and $y=y_1$; but the left side is also zero when $x=x_1$ and $y=y_1$, and therefore the straight line passes through (x_1, y_1). Similarly it may be proved to pass through (x_2, y_2).

Equation (7) when simplified is the same as (5); if in (7) we put $x_2=x_1$ and $y_2=y_1$, we get equation (6).

When the equation of the circle is

$$x^2+y^2+2gx+2fy+c=0, \quad \text{..............(1')}$$

we find for the gradient, instead of equation (4), the equation

$$\frac{y_1-y_2}{x_1-x_2}=-\frac{x_1+x_2+2g}{y_1+y_2+2f}, \quad \text{................(4')}$$

and then, instead of equation (5), we obtain

$$\begin{aligned}(x_1+x_2+2g)x+(y_1+y_2+2f)y \\ &=x_1^2+y_1^2+2gx_1+2fy_1+x_1x_2+y_1y_2 \\ &=-c+x_1x_2+y_1y_2. \quad \text{..........................(5')}\end{aligned}$$

We then deduce the equation of the tangent

$$x_1x+y_1y+g(x+x_1)+f(y+y_1)+c=0. \quad \text{........(6')}$$

Instead of equation (7), we take now

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=x^2+y^2+2gx+2fy+c, \quad (7')$$

which is readily seen to give the secant through (x_1, y_1),

(x_2, y_2). Put $x_2 = x_1$ and $y_2 = y_1$, and we get the equation of the tangent.

55. Coincident Points. The idea of coincident points may be utilised in other ways when treating of problems on tangents. For example, consider the equations

$$y = 2x + c, \ldots\ldots (1) \qquad x^2 + y^2 = 20. \ldots\ldots (2)$$

The line (1) intersects the circle (2) in points whose coordinates are obtained by solving (1) and (2) as simultaneous equations. In equation (2) put $2x + c$ for y, and we find for the abscissae of the points of intersection the equation

$$5x^2 + 4cx + c^2 - 20 = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Equation (3), being a quadratic, gives two values, x_1 and x_2 say, for x, and then equation (1) gives two corresponding values y_1 and y_2 for y; the line is therefore, in general, a secant which cuts the circle at the points (x_1, y_1) and (x_2, y_2). The values of x_1, y_1, x_2, y_2 are

$$x_1 = \frac{-2c + \sqrt{(100 - c^2)}}{5}; \quad x_2 = \frac{-2c - \sqrt{(100 - c^2)}}{5};$$

$$y_1 = \frac{c + 2\sqrt{(100 - c^2)}}{5}; \qquad y_2 = \frac{c - 2\sqrt{(100 - c^2)}}{5}.$$

If $c^2 < 100$ these values are real and unequal, and the secant cuts the circle in two real and distinct points.

If $c^2 > 100$ the values are imaginary and the line does not meet the circle at all; but just as equation (3) is said to have two *imaginary* roots rather than to have no roots, so it is convenient to say in this case that the line intersects the circle in two *imaginary points.* The conception of imaginary points of intersection often simplifies the statement of theorems.

There is, however, another case, namely $c^2 = 100$. Equation (3) is still a quadratic, but its roots are now equal and the points (x_1, y_1), (x_2, y_2) are *coincident,* the point in which they coincide being

$$\left(-\frac{2c}{5}, \frac{c}{5}\right);$$

the line (1) is now a tangent and $(-2c/5, c/5)$ is its point of contact. The solution $x=-2c/5$ and $y=c/5$ may be called a *repeated* solution, since $x=-2c/5$ *twice* and $y=c/5$ *twice*.

When $c^2=100$, we have $c=10$ or -10; we thus have two tangent lines.

When $c=10$, the solution $x=-4$ and $y=2$ is a *repeated* solution, and the line (1) becomes

$$y=2x+10,$$

which touches the circle (2) at $(-4, 2)$.

When $c=-10$, the solution $x=4$ and $y=-2$ is a *repeated* solution, and the line (1) becomes

$$y=2x-10,$$

which touches the circle (2) at $(4, -2)$.

We have thus solved the problem of finding the tangent to the circle (2) of gradient 2; there are two solutions, as is geometrically obvious.

Again, consider the question: what relation must hold between the constants m and c if the line $y=mx+c$ is a tangent to the circle $x^2+y^2=r^2$?

The equation for the abscissae of the points in which the line cuts the circle is

$$(1+m^2)x^2+2cmx+c^2-r^2=0.$$

The two points will be coincident, and the line will therefore be a tangent if this equation have equal roots. But the condition for equal roots is

$$4c^2m^2=4(1+m^2)(c^2-r^2) \quad \text{or} \quad c^2=r^2(1+m^2).$$

Thus the line $y=mx+r\sqrt{(1+m^2)}$ is a tangent whatever be the value of m, and since the root may have either the positive or the negative sign there are two tangents for any one value of m.

Ex. 1. Find the equation of the tangents from the point (7, 9) to the circle

$$x^2+y^2=13, \qquad \text{.....(i)}$$

and state the coordinates of their points of contact.

The equation of any line through (7, 9) is of the form

$$y-9=m(x-7) \quad \text{or} \quad y=mx+(9-7m). \qquad \text{.....(ii)}$$

The abscissae of the points in which the line and circle intersect are given by the equation

$$(1+m^2)x^2+2m(9-7m)x+(9-7m)^2-13=0, \quad \text{.........(iii)}$$

and the roots of this equation are equal if

$$4m^2(9-7m)^2=4(1+m^2)\{(9-7m)^2-13\},$$

that is, if $$36m^2-126m+68=0,$$

that is, if $$m=\tfrac{2}{3} \quad \text{or} \quad \tfrac{17}{6}.$$

When $m=\frac{2}{3}$ equation (ii) becomes $y=\frac{2}{3}x+\frac{13}{3}$, which is one tangent. To find its point of contact, note that when $m=\frac{2}{3}$ equation (iii) gives $x=-2$ twice, and then (ii) gives $y=3$ twice, so that the point of contact is $(-2, 3)$.

When $m=\frac{17}{6}$ the tangent is $y=\frac{17}{6}x-\frac{65}{6}$, and the point of contact is $(\frac{17}{5}, -\frac{6}{5})$.

Ex. 2. Show that $y=x-1$ is a tangent to the circle

$$x^2+y^2-8x-2y+15=0,$$

and find the coordinates of its point of contact.

Solving these equations as simultaneous equations, we find for the abscissae the equation

$$x^2-6x+9=0,$$

that is, $$(x-3)(x-3)=0.$$

The two values of x, and therefore also, since $y=x-1$, the two values of y, are equal. The line is thus a tangent, and $(3, 2)$ is the point of contact.

Ex. 3. Show that the tangent at the origin to the circle

$$x^2+y^2+2gx+2fy=0$$

is $gx+fy=0$.

If these equations be solved as simultaneous equations we see that the solutions are $x=0$ twice, $y=0$ twice; the line therefore meets the circle in two coincident points and is therefore a tangent.

Ex. 4. Find the relation between the constants of the equation

$$x^2+y^2+2gx+2fy+c=0$$

if the x-axis is a tangent to the circle.

The circle meets the x-axis where

$$x^2+2gx+c=0;$$

if the x-axis is a tangent, the two roots of this equation must be equal and therefore $c=g^2$. This is the required relation, and

$$x^2+y^2+2gx+2fy+g^2=0$$

is the equation of a circle which touches the x-axis at $(-g, 0)$.

Ex. 5. Find the equations of the tangents to the circle

$$x^2+y^2-6x-8y+23=0$$

that are parallel to the line $x+y=0$, and give the coordinates of the points of contact.

The two tangents are

$$x+y=5 \quad \text{and} \quad x+y=9,$$

and the points of contact are (2, 3) and (4, 5).

Ex. 6. For what values of a will the circle

$$x^2+y^2-2ax-4=0$$

have the line $x=2y-6$ as a tangent?

The ordinates of the points of intersection are given by the equation

$$(2y-6)^2+y^2-2a(2y-6)-4=0$$

or

$$5y^2-2(2a+12)y+(12a+32)=0;$$

the points of intersection will be coincident if this equation in y has equal roots, that is, if

$$(2a+12)^2=5(12a+32),$$

or if $(a-4)(a+1)=0$, or finally if $a=4$ or -1.

The line $x=2y-6$ is therefore a tangent to each of the circles

$$x^2+y^2-8x-4=0, \quad x^2+y^2+2x-4=0.$$

The points of contact are (2, 4) and $(-2, 2)$ respectively.

56. The Square on the Tangent from a Point. Let $P(x_1, y_1)$ be a given point outside the given circle

$$x^2+y^2+2gx+2fy+c=0$$

whose centre is C; it is required to find an expression for the square of the tangent PT from P to the circle.

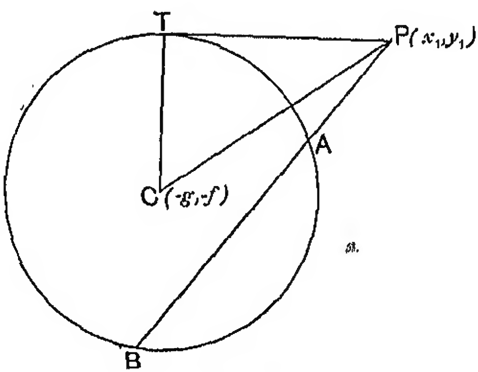


FIG. 43.

The angle PTC (Fig. 43) is a right angle, so that

$$PT^2=CP^2-CT^2.$$

Now

$$CP^2 = \text{square of distance between } (x_1, y_1) \text{ and } (-g, -f)$$
$$= (x_1+g)^2+(y_1+f)^2$$
$$= x_1^2+y_1^2+2gx_1+2fy_1+g^2+f^2;$$
$$CT^2 = \text{square of radius of circle}$$
$$= g^2+f^2-c;$$

so that $\quad CP^2-CT^2 = x_1^2+y_1^2+2gx_1+2fy_1+c.$

Hence: *the square of the tangent from* (x_1, y_1) *to the circle*

$$x^2+y^2+2gx+2fy+c=0$$

is

$$\mathbf{x_1^2+y_1^2+2gx_1+2fy_1+c.}$$

If a secant PAB through P cut the circle in A, B (Fig. 43), then

$$PA \, . \, PB = PT^2 = x_1^2+y_1^2+2gx_1+2fy_1+c.$$

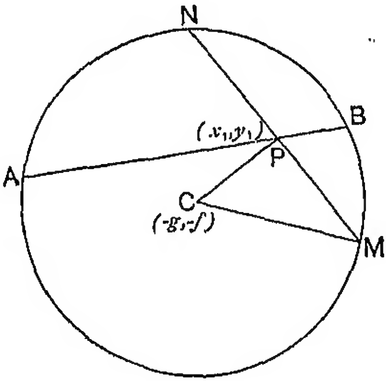


FIG. 44.

If P lie within the circle (Fig. 44) and a secant PAB be drawn to cut the circle in A, B, and also the chord MPN be drawn perpendicular to CP, then (attending to sign)

$$PA \, . \, PB = PM \, . \, PN = -PM^2 = -(CM^2-CP^2)$$
$$= CP^2-CM^2$$
$$= (x_1+g)^2+(y_1+f)^2-(g^2+f^2-c)$$
$$= x_1^2+y_1^2+2gx_1+2fy_1+c.$$

When P is within the circle $x_1^2+y_1^2+2gx_1+2fy_1+c$ is negative; when P is on the circle $x_1^2+y_1^2+2gx_1+2fy_1+c$ is zero; when P is without the circle $x_1^2+y_1^2+2gx_1+2fy_1+c$ is positive.

The boundary for which this important expression vanishes separates the region in which it is positive from the region in which it is negative. This is an example of a *general* principle of sign; for instance ax_1+by_1+c changes from positive to negative as (x_1, y_1) crosses the boundary line $ax+by+c=0$.

If then a secant through a point $P(x_1, y_1)$ to the circle

$$x^2+y^2+2gx+2fy+c=0$$

cut the circle in A and B, the product $PA \cdot PB$ is equal in sign and magnitude to

$$x_1^2+y_1^2+2gx_1+2fy_1+c;$$

and $PA \cdot PB$ is known as the **power of the point** P with respect to the circle. When P is outside, the power of the point is equal to the square on the tangent from P; and indeed the phrase "square on the tangent from a point" is commonly used instead of "the power of a point," even when the point is inside the circle.

COR. The square on the tangent from $P(x_1, y_1)$ to the circle $Ax^2+Ay^2+2Gx+2Fy+C=0$ is

$$x_1^2+y_1^2+2\frac{G}{A}x_1+2\frac{F}{A}y_1+\frac{C}{A}.$$

Ex. 1. Find the square of the tangent from (2, 1) to $x^2+y^2-1=0$.

Ex. 2. Find the square of the tangent from (1, 3) to

$$x^2+y^2-2x-2y+1=0.$$

Ex. 3. Find the length of the tangent to the circle

$$2x^2+2y^2-x+3y+1=0$$

from $(1, -1)$; and show that the other point on the line $x+2y+1=0$, from which a tangent to this circle has the same length, is $(-2/5, -3/10)$.

Ex. 4. Prove that the lengths of the tangents to the two circles $x^2+y^2+2x-4=0$ and $x^2+y^2-3x-4=0$ from (0, 5) are equal.

Ex. 5. Prove that the point (1, 2) is tangentially equidistant from the two circles

$$x^2+y^2+2x+3y+1=0, \quad x^2+y^2+x+2y+4=0.$$

Ex. 6. Prove that all points on the y-axis are tangentially equidistant from

$$x^2+y^2-x+4=0 \quad \text{and} \quad x^2+y^2+5x+4=0.$$

Ex. 7. Prove that all points on the line $x+y+1=0$ are tangentially equidistant from the circles

$$x^2+y^2+7x-y+5=0 \quad \text{and} \quad x^2+y^2+6x-2y+4=0.$$

EXERCISES XVI.

1. Find the equation of the circle which touches the x-axis at the point $(4, 0)$ and passes through the point $(0, 2)$ on the y-axis. At what other point does the circle intersect the y-axis?

2. Find the equation of the circle which touches the y-axis at the point $(0, 3)$ and passes through the point $(2, 5)$. What is the equation of the tangent at $(2, 5)$?

3. What is the equation of the circle which touches the x-axis at the point $(a, 0)$ and also touches the line $y=b$?

4. Find the equations of the circles which touch the x-axis at the point $(3, 0)$ and also touch the line $3y-4x=12$.

5. Find the equations of the circles which touch the coordinate axes and the line $3x+4y=12$.

6. M is the projection of a point P on the line $x+13=0$ and T is the point of contact of a tangent from P to the circle $x^2+y^2=25$; if $PT^2=4MP$, find the equation of the locus of P and draw the locus.

7. M is the projection of a point P on the line $x+a=0$ and T is the point of contact of a tangent from P to the circle $x^2+y^2=r^2$; if $PT^2=2p \cdot MP$, where $2p$ is a given length, find the equation of the locus of P and draw the locus.

8. M and N are the points of contact of tangents from P to two circles whose centres are $(0, 0)$ and $(c, 0)$ and whose radii are a and b respectively; if P moves so that PM is to PN as a is to b $(a \neq b)$, show that the locus of P is a circle and draw the circle.

9. If the tangents from P to two concentric circles are inversely as the radii of the circles, show that the locus of P is a concentric circle.

10. A point P moves so that the length of the tangent from it to the circle
$$x^2+y^2-2ax+p=0$$
is k times the length of the tangent from it to the circle
$$x^2+y^2-2bx+p=0;$$
show that the locus of P is a circle. Draw the circles for the case $a=-7$, $b=5$, $p=9$, $k=2$.

11. Find the equations of the common tangents to the circles whose equations are

$$x^2+y^2=25, \quad (x-12)^2+y^2=9.$$

12. Show that one pair of common tangents to the circles whose equations are ($a^2 > b > 0$)

$$x^2+y^2-2ax+b=0, \quad x^2+y^2-2kax+k^2b=0$$

goes through the origin.

If these circles cut the x-axis at A, B and A', B' respectively, show that $OA \,.\, OB' = OA' \,.\, OB$, where A is the point of the first circle and A' the point of the second circle nearest to the origin O.

13. The line joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ cuts the circle $x^2+y^2=r^2$ at A and B; show that the ratios $PA : AQ$ and $PB : BQ$ are the values of the ratio $m : n$ given by the equation

$$m^2(x_2^2+y_2^2-r^2)+2mn(x_1x_2+y_1y_2-r^2)+n^2(x_1^2+y_1^2-r^2)=0.$$

If PQ is a tangent to the circle, then

$$(x_1x_2+y_1y_2-r^2)^2=(x_1^2+y_1^2-r^2)(x_2^2+y_2^2-r^2).$$

Deduce that the equation of the pair of tangents from P to the circle is

$$(x_1x+y_1y-r^2)^2=(x_1^2+y_1^2-r^2)(x^2+y^2-r^2).$$

CHAPTER VIII.

COAXAL CIRCLES. POLE AND POLAR.

57. Radical Axis. Definition: If a variable point move so that the squares on the tangents from it to two circles are equal, the locus of the point is called the radical axis of the circles. Note that the phrase "square on the tangent from a point to a circle" is to be understood in the sense explained at the end of § 56.

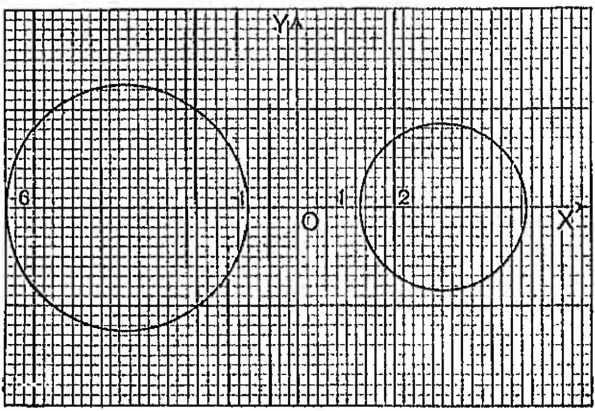


FIG. 45.

Let $$x^2+y^2+2gx+2fy+c=0 \quad\text{...............(1)}$$
and $$x^2+y^2+2g'x+2f'y+c'=0 \quad\text{...............(2)}$$
represent any two given circles; it is required to find the radical axis of the two circles.

Let (h, k) be a point on the radical axis.

Then the square of the tangent from (h, k) to (1) is

$$h^2+k^2+2gh+2fk+c;$$

and the square of the tangent from (h, k) to (2) is

$$h^2+k^2+2g'h+2f'k+c'.$$

Therefore

$$h^2+k^2+2gh+2fk+c=h^2+k^2+2g'h+2f'k+c',$$

that is, $$2(g-g')h+2(f-f')k+(c-c')=0.$$

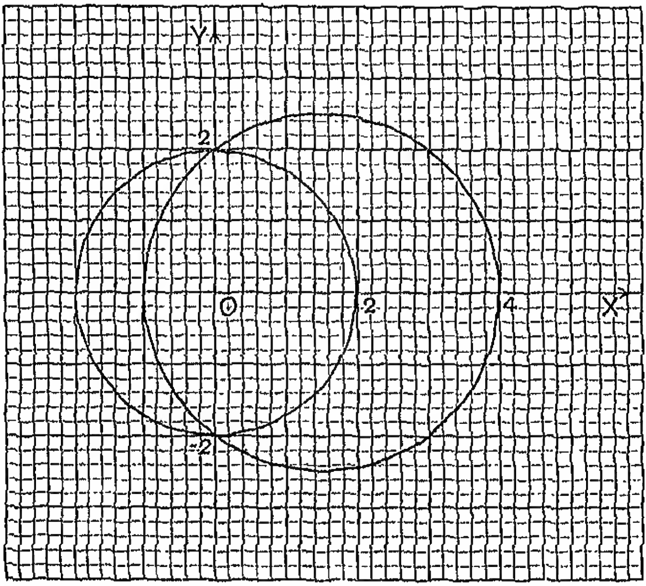


FIG. 46.

Writing (x, y) for (h, k) to denote the coordinates of *any* point on the radical axis, we get

$$2(g-g')x+2(f-f')y+(c-c')=0$$

as the equation of the radical axis.

The radical axis is therefore a straight line perpendicular to the line of centres.

For example, draw the two circles

$$x^2+y^2+7x+6=0 \quad \text{and} \quad x^2+y^2-6x+6=0 \text{ (Fig. 45).}$$

The radical axis is given by the equation

$$x^2+y^2+7x+6=x^2+y^2-6x+6,$$

that is, $$x=0.$$

Again draw the two circles

$$x^2+y^2-4=0, \quad x^2+y^2-3x-4=0 \text{ (Fig. 46).}$$

The radical axis is

$$x^2+y^2-4=x^2+y^2-3x-4,$$

that is, $$x=0.$$

Note. When two circles intersect their common chord is the radical axis; because each point of intersection lies on the radical axis, and the radical axis is a straight line. Even when the circles do not intersect, the radical axis is a real line and is still called the common chord.

EXERCISES XVII.

1. Prove that the radical axis of the circles

$$x^2+y^2-7y+6=0 \quad \text{and} \quad x^2+y^2-5y+6=0$$

is the x-axis. Draw the figures.

2. Find the radical axis of

(i) $x^2+y^2+3x-y+2=0$ and $x^2+y^2+2x-y-3=0$;

(ii) $x^2+y^2-2x-3y=5$ and $x^2+y^2-7x+2y-4=0$;

(iii) $x^2+y^2-3x+2y-4=0$ and $2x^2+2y^2-x+y-1=0$;

(iv) $3x^2+3y^2-4x-6y-1=0$ and $2x^2+2y^2-3x-2y-4=0$.

3. If N is the foot of the perpendicular from any point P to the radical axis of the two circles, centres A and B, whose equations are

$$x^2+y^2-px-a=0, \quad x^2+y^2-qx-a=0,$$

prove that the difference of the squares on the tangents from P to the two circles is $2AB\,.\,NP$.

4. Prove that the radical axes of three circles, taken in pairs, are concurrent. The point of concurrence is called the radical centre.

5. Prove that the three circles

$$x^2+y^2-4=0, \quad x^2+y^2-3x-4=0, \quad x^2+y^2+5x-4=0$$

all pass through the points $(0, 2)$ and $(0, -2)$. Draw the circles, and find their common radical axis.

raw the circles

$$+y^2-5x+6=0,\quad x^2+y^2+7x+6=0,\quad x^2+y^2-9x+6=0.$$

that they have a common radical axis. What two imaginary
the y-axis are common to all the circles?

raw the circles represented by the equation

$$x^2+y^2-ax-4=0,$$

, 1, -2, 3, -4. Prove that they all pass through two fixed
nd find the coordinates of the points.

raw the circles represented by the equation

$$x^2+y^2-ax+4=0,$$

, -4, 5, -6. Prove that they all pass through two fixed
ary) points, and find the coordinates of the points.

raw the system of circles represented by the equation

$$x^2+y^2-5x-6+k(x^2+y^2+x-6)=0,$$

ing values of the constant k. Prove that they all go through
ts of intersection of

$$x^2+y^2-5x-6=0 \quad\text{and}\quad x^2+y^2+x-6=0.$$

line is their common radical axis?

Draw the system of circles represented by the equation

$$x^2+y^2-4x+k(x^2+y^2+2x)=0,$$

g various values of the constant k. What line is the common
axis of the system?

Draw the system of circles represented by the equation

$$x^2+y^2-9x+8+k(x^2+y^2+6x+8)=0,$$

g various values of the constant k. Through what two fixed
ary) points do all the circles pass? What line is the common
axis?

Coaxal Circles. A system of circles, every member
ch passes through two fixed points, is called a coaxal
of circles. The line joining the two fixed points is
dical axis of every pair of the circles.

The equation

$$x^2+y^2-ax-b=0, \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

b is a fixed constant and a a varying constant or
neter (§ 38) represents a system of coaxal circles,
all pass through the two fixed points $(0, \sqrt{b})$ and
$/b)$. These points are real and distinct if b is a
ve number; they are real and coincident if b is zero;

they are imaginary if b is negative. The common radical axis is the y-axis.

Fig. 47 represents the system for the case in which $b=9$ and a has the values 0, 2, -4, 6, -7.

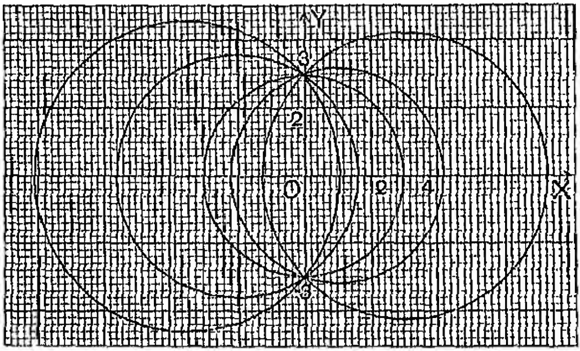


FIG. 47.

Fig. 48 represents the system for the case in which $b=-4$ and a has the values 5, 6, 7, 8, -5, -6, -7, -8.

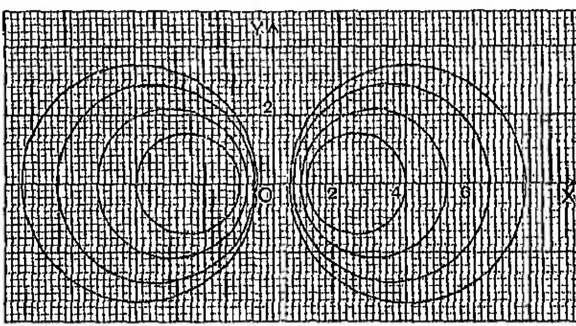


FIG. 48.

When b is negative, say $b=-c^2$, equation (1) may be written

$$(x-\tfrac{1}{2}a)^2+y^2=\frac{a^2}{4}-c^2. \qquad (2)$$

When $a=2c$ the radius of the circle is zero; the circle

has become the point $(c, 0)$. Similarly, when $a = -2c$ the circle reduces to the point $(-c, 0)$. These two points $(c, 0), (-c, 0)$ are called the **limiting points** of the system of coaxal circles given by (1) when b is negative and equal to $-c^2$. The points $(2, 0), (-2, 0)$ are the limiting points of the system represented in Fig. 48. Evidently the limiting points of the system given by equation (1) are real when, and only when, b is negative.

II. The equation

$$x^2+y^2+2gx+2fy+c+k(x^2+y^2+2g'x+2f'y+c')=0,$$

where k is a varying constant and g, f, c, g', f', c' fixed constants, represents a system of coaxal circles which pass through the fixed points in which the fixed circles

$$x^2+y^2+2gx+2fy+c=0, \quad x^2+y^2+2g'x+2f'y+c'=0$$

intersect. (Compare § 38.)

Ex. 1. The equation $x^2+y^2-ax-4=0$ represents a system of coaxal circles; find the equation of the circle of the system which passes through the point $(1, 2)$.

Ex. 2. Find the equation of the circle coaxal with

$$x^2+y^2-7x+12=0 \quad \text{and} \quad x^2+y^2+8x+12=0$$

which passes through the point $(-2, 3)$.

Ex. 3. The equation $x^2+y^2-ax-9=0$ represents a system of coaxal circles; find the equations of the circles of the system which touch the line $x+3y=11$.

Ex. 4. Find the equation of the circle coaxal with

$$x^2+y^2-2x+2y+1=0 \quad \text{and} \quad x^2+y^2+8x-6y=0$$

which passes through $(-1, -2)$.

Ex. 5. Find the equations of the circles coaxal with

$$x^2+y^2-6x+4=0 \quad \text{and} \quad x^2+y^2+5x+4=0$$

which touch the line $3x-4y=15$.

59. Orthogonal Circles. Let P be any point on a circle, centre A (Fig. 49), and let B be any point on the tangent at P. With B as centre and radius BP describe a second circle. The radii AP, BP to the point of intersection of the circles are at right angles; the two circles are said to cut *orthogonally* at P.

If d denote the distance AB between the centres A and B of two orthogonal circles of radii a and b, then clearly

$$d^2 = a^2 + b^2,$$

and conversely.

Let $$x^2 + y^2 + 2gx + 2fy + c = 0$$

and $$x^2 + y^2 + 2g'x + 2f'y + c' = 0$$

be two circles; it is required to find the condition that they be orthogonal circles.

Let d = distance between centres.

„ a^2 = square of radius of first circle.

„ b^2 = square of radius of second circle.

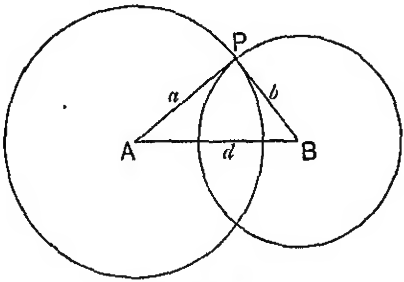


FIG. 49.

Now the coordinates of the centres are $(-g, -f)$ and $(-g', -f')$.

Hence $$d^2 = (g-g')^2 + (f-f')^2.$$

Also $$a^2 = g^2 + f^2 - c$$

and $$b^2 = g'^2 + f'^2 - c'.$$

The condition that the circles be orthogonal is

$$d^2 = a^2 + b^2,$$

that is, $$(g-g')^2 + (f-f')^2 = g^2 + f^2 - c + g'^2 + f'^2 - c'$$

or $$2gg' + 2ff' = c + c'.$$

Ex. 1. Prove that the circles

$$x^2 + y^2 = 4 \quad \text{and} \quad x^2 + y^2 - 5x + 4 = 0$$

are orthogonal.

2. Prove that the circles

$$x^2+y^2=4 \quad\text{and}\quad x^2+y^2-ax+4=0$$

hogonal.

3. Prove that every circle through the points (2, 0) and (−2, 0) ogonal to every circle of the system $x^2+y^2-ax+4=0$.

4. Prove that the circle $x^2+y^2-ax+b^2=0$ is orthogonal to the through the points $(b, 0)$, $(-b, 0)$, $(0, c)$.

5. Find the equation of the circle orthogonal to the two circles

$$x^2+y^2-9x+14=0,\quad x^2+y^2+15x+14=0,$$

ssing through the point (2, 5).

6. Give geometrical solutions of the questions in Exs. 1–5.

7. Prove that every circle of the system

$$x^2+y^2-2ax+d^2=0$$

hogonal to each circle of the system

$$x^2+y^2-2by-d^2=0,$$

a and b are varying constants. Draw diagrams of the two ns referred to the same axes.

. **Inverse Points. Definition.** **If O is the centre of a of radius r, and P and P′ two points lying on a line**

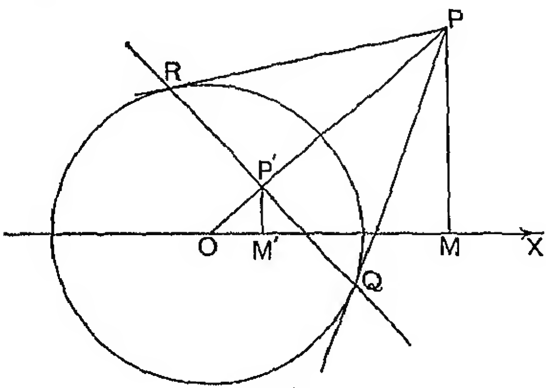


FIG. 50.

ugh O such that OP . OP′ = r², then P and P′ are called rse points with respect to the circle. The constant OP' is sometimes called the constant of inversion.

P, P' are inverse points with respect to a circle, radius r, whose centre is the origin, and (x, y) are the coordinates of P; to find the coordinates (x', y') of P'.

Let M, M' be the projections of P, P' on $X'OX$ (Fig. 50).

Then $$\frac{OM'}{OM}=\frac{OP'}{OP}=\frac{OP.OP'}{OP^2}.$$

But $OM'=x'$, $OM=x$, $OP.OP'=r^2$, $OP^2=x^2+y^2$.

Therefore $$\frac{x'}{x}=\frac{r^2}{x^2+y^2} \quad \text{or} \quad x'=\frac{r^2x}{x^2+y^2}.$$

Similarly, $$\frac{y'}{y}=\frac{M'P'}{MP}=\frac{OP'}{OP}=\frac{OP.OP'}{OP^2}=\frac{r^2}{x^2+y^2}$$

or $$y'=\frac{r^2y}{x^2+y^2}.$$

We may also show that

$$x=\frac{r^2x'}{x'^2+y'^2} \quad \text{and} \quad y=\frac{r^2y'}{x'^2+y'^2}.$$

If P (x, y) and P' (x', y') are inverse points with respect to the circle
$$x^2+y^2+2gx+2fy+c=0,$$
then $r^2=g^2+f^2-c$, O is the point $(-g, -f)$, and
$$OM'=x'+g,\ OM=x+g,\ OP^2=(x+g)^2+(y+f)^2,$$
so that $$\frac{x'+g}{x+g}=\frac{g^2+f^2-c}{(x+g)^2+(y+f)^2}=\frac{y'+f}{y+f}.$$

Ex. 1. Find the coordinates of the point inverse to (2, 3) with respect to $x^2+y^2=1$.

Ex. 2. If a point P trace out the straight line $x=2$, find the equation of the locus traced out by P', the inverse of P with respect to the circle $x^2+y^2=1$.

Ex. 3. If a point P trace out the straight line $x=a$, find the equation of the locus of P', the inverse of P with respect to $x^2+y^2=k^2$.

Ex. 4. If a point P trace out the circle whose equation is
$$x^2+y^2-2x=0,$$
find the equation of the locus of P', the inverse of P with respect to
$$x^2+y^2=6.$$

. If a point P trace out the circle $x^2+y^2-4x+3=0$, find the P', the inverse of P with respect to the circle $x^2+y^2=12$.

. If P, P' are inverse points with respect to a circle, prove ery circle through P and P' is orthogonal to the given circle.

. If a point P trace out the circle

$$(x-a)^2+y^2=r^2,$$

hat the inverse of P with respect to the circle $x^2+y^2=k$ traces circle

$$x^2+y^2-\frac{2ak}{a^2-r^2}x+\frac{k^2}{a^2-r^2}=0.$$

Pole and Polar. Definition. **The perpendicular to the OP through P′, the inverse of P with respect to a centre O, is called the polar of P with respect to the**

; P (Fig. 50, p. 133) be the point (x_1, y_1); to find the ion of the polar of P with respect to the circle whose ion is

$$x^2+y^2=r^2.$$

e gradient of OP (where O is the origin) is $\frac{y_1}{x_1}$.

nce the gradient of the polar $=-\frac{x_1}{y_1}$.

e coordinates of P', the inverse of P, are

$$\frac{r^2x_1}{x_1^2+y_1^2},\quad \frac{r^2y_1}{x_1^2+y_1^2}.$$

nce the polar of (x_1, y_1) is the line through

$$\left(\frac{r^2x_1}{x_1^2+y_1^2},\ \frac{r^2y_1}{x_1^2+y_1^2}\right) \text{ of gradient } -\frac{x_1}{y_1}.$$

herefore the equation of the polar is

$$y-\frac{r^2y_1}{x_1^2+y_1^2}=-\frac{x_1}{y_1}\left(x-\frac{r^2x_1}{x_1^2+y_1^2}\right),$$

is,

$$xx_1+yy_1=\frac{r^2x_1^2}{x_1^2+y_1^2}+\frac{r^2y_1^2}{x_1^2+y_1^2}$$

$$\mathbf{xx_1+yy_1=r^2.}$$

he point $P(x_1, y_1)$ is called the pole of the line

$$xx_1+yy_1=r^2.$$

Similarly, it may be shown that the polar of (x_1, y_1) with respect to

$$x^2+y^2+2gx+2fy+c=0$$

is $$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.$$

It is important to notice that if (x_1, y_1) lies *without* a circle, its polar is the chord of contact of the tangents drawn from the point to the circle; QR in Fig. 50 is the polar of P.

Hence, if (x_1, y_1) lies without the circle

$$x^2+y^2=r^2,$$

the equation of the chord of contact of tangents from (x_1, y_1) is

$$xx_1+yy_1=r^2;$$

if (x_1, y_1) lies without the circle

$$x^2+y^2+2gx+2fy+c=0,$$

the equation of the chord of contact is

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.$$

62. The Polar as a Locus. Let any secant of the circle $x^2+y^2=r^2$ through the point $P(x_1, y_1)$ meet the circle in A and B, and let the tangents at A and B meet in Q; to prove that the locus of Q is the polar of P.

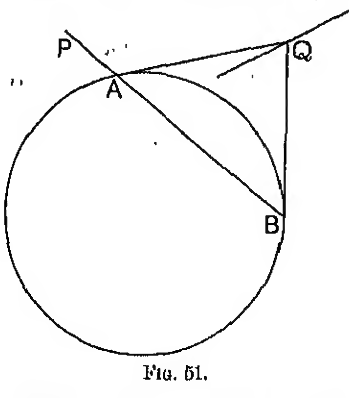


FIG. 51.

Let Q be the point (h, k) (Fig. 51).

Then AB is the chord of contact of tangents from (h, k).

Therefore the equation of AB is

$$xh+yk=r^2.$$

But $P(x_1, y_1)$ lies on this line; therefore

$$hx_1+ky_1=r^2.$$

Writing x, y for h, k to denote a variable point Q, we get

$$xx_1+yy_1=r^2$$

as the equation of the locus of Q.

But this is the equation of the polar of P.

Hence the locus of Q is the polar of P.

In Chapter XXII. the polar is discussed from a different point of view by methods which are applicable to the circle. See also Exercises XVIII., Example 41.

63. Reciprocal Property of Pole and Polar. THEOREM. *If a point A lies on the polar of B with respect to a circle, then B lies on the polar of A* (Fig. 52).

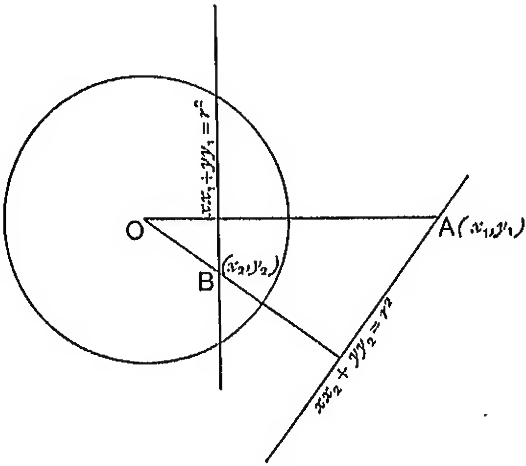


FIG. 52.

Draw rectangular axes of reference $X'OX$, $Y'OY$ through O, the centre of the circle. Let r be the radius of the circle. Let the coordinates of A and B, referred to the axes, be (x_1, y_1) and (x_2, y_2) respectively.

Then the polar of B is the line

$$xx_2+yy_2=r^2.$$

But $A(x_1, y_1)$ lies on the polar of B;

therefore $$x_1x_2+y_1y_2=r^2. \quad\text{.......................(1)}$$

Now the polar of A is the line

$$xx_1+yy_1=r^2.$$

Hence $B(x_2, y_2)$ lies on the polar of A if

$$x_2x_1 + y_2y_1 = r^2;$$

and this is true by (1). Hence the theorem is established.

Points such that the polar of each passes through the other are called **conjugate points**; the polars are called **conjugate lines.**

Ex. 1. Find the equation of the polar of (2, 5) with respect to the circle $x^2+y^2=1$.

Ex. 2. Find the equation of the polar of (0, 0) with respect to the circle $x^2+y^2-4x-2y=4$.

Ex. 3. Tangents to the circle $x^2+y^2=4$ are drawn at the points where the circle meets the line $x+y=1$. Find the coordinates of their point of intersection.

Ex. 4. Find the equation of the chord of contact of tangents drawn from (2, 1) to the circle $x^2+y^2+2x+3y=4$.

Ex. 5. From $P(x_1, y_1)$ and $Q(x_2, y_2)$ are drawn perpendiculars PM, QN to the polars of Q and P with respect to the circle $x^2+y^2=r^2$; prove that $OP/PM = OQ/QN$, O being the origin.

Ex. 6. Find the coordinates of the point of intersection of the polars of (3, 2) with respect to the circles

$$x^2+y^2-7x+10=0 \quad \text{and} \quad x^2+y^2+11x+10=0.$$

Ex. 7. If P is the point (1, 7) and Q the intersection of the polars of P with respect to the circles

$$x^2+y^2-8x+12=0 \quad \text{and} \quad x^2+y^2+7x+12=0,$$

prove that the circle on PQ as diameter is orthogonal to the two given circles.

Ex. 8. Prove that the polars of (x_1, y_1) with respect to the system of coaxal circles specified by the equation

$$x^2+y^2-2ax+c=0,$$

where a is a varying constant, all pass through the fixed intersection of the lines

$$xx_1+yy_1+c=0 \quad \text{and} \quad x+x_1=0.$$

EXERCISES XVIII.

1. If $y=mx+2\frac{1}{2}$ touches $x^2+y^2=4$, find the values of m and illustrate by a figure.

2. Prove that $3x+5y=34$ touches $x^2+y^2=34$, and find the coordinates of the point of contact.

3. Find the equation of the line which touches $x^2+y^2=8x+12y$ at the point (8, 0).

4. Find the coordinates of the points at which the straight line $y=x+2$ cuts $x^2+y^2=2(x+4)$. Find also the equations of the tangents at these points and the coordinates of their point of intersection.

5. Find the equations of the straight lines parallel to $x+y=0$ which touch $x^2+y^2=8$. Illustrate by a figure.

6. Prove that

$$x+1=0, \quad y=5, \quad 3x+4y=11, \quad 3y=4x+7$$

are the four common tangents to the circles

$$x^2+y^2-8y+15=0$$

and
$$x^2+y^2+8x-4y+11=0.$$

7. Show that the angle between the tangents drawn from the point (3, 4) to the circle $x^2+y^2-2x-4y+4=0$ is $\cos^{-1}\frac{3}{5}$.

Also show that $y-x\cot\phi=2+\tan\dfrac{\phi}{2}$ touches this circle for all values of ϕ.

8. Find the equation of the common chord of the circles

$$(x-a)^2+y^2=a^2\,; \quad x^2+(y-b)^2=b^2.$$

Also find the length of the common chord, and show that the circle described on the common chord as diameter is

$$(a^2+b^2)(x^2+y^2)=2ab(bx+ay).$$

9. Prove that the length of the common chord of the two circles

$$x^2+y^2-2px+b^2=0 \quad \text{and} \quad x^2+y^2-2qy-b^2=0$$

is
$$2\sqrt{\{(p^2-b^2)(q^2+b^2)/(p^2+q^2)\}}.$$

10. The straight line $3x-y=2$ meets the lines $y=x$ and $y=2x$ in P and Q. Find the equation of the circle on PQ as diameter.

11. The equation $ax+by=c$ represents a line which cuts the circle $x^2+y^2=r^2$ in A and B. Prove that the coordinates of the middle point of AB are $ac/(a^2+b^2)$, $bc/(a^2+b^2)$.

12. Show that $x=a\cos\theta$, $y=a\sin\theta$ are the coordinates of a point on the circle $x^2+y^2=a^2$ for every value of θ.

If the extremities of a chord of the circle are $(a\cos\theta, a\sin\theta)$ and $(a\cos\phi, a\sin\phi)$, prove that the equation of the chord is

$$x\cos\frac{\theta+\phi}{2}+y\sin\frac{\theta+\phi}{2}=a\cos\frac{\theta-\phi}{2}\,;$$

and deduce the equation of the tangent at the first point.

13. Find the equation of that chord of the circle $x^2+y^2=8$ which is bisected at the point $(-1, 2)$.

14. Tangents TP and TQ are drawn from $T(x_1, y_1)$ to the circle $x^2+y^2=r^2$; find the equation of the circle circumscribing the triangle TPQ.

15. Trace the loci whose equations are

$$2x+y=3\ ; \quad x^2+y^2=2\ ; \quad (x-2)^2+(y-1)^2=1.$$

Find the two points common to the three loci.

16. Find the square of the tangent from (x_1, y_1) to the circle

$$a(x^2+y^2)+2gx+2fy+c=0.$$

17. The square of the tangent from P to the circle $x^2+y^2-8x+4=0$ is minus the square of the distance from P to the point $(2, 0)$; prove that P moves on a circle whose centre bisects the line drawn from the given point to the centre of the given circle.

18. A and B are the centres of the circles

$$x^2+y^2-2px+b^2=0\ ; \quad x^2+y^2+2qx+b^2=0.$$

A point P moves so that the ratio of the squares of the tangents from it to those circles is m/n; prove that it describes a circle whose centre C divides AB so that $AC/BC=m/n$.

19. Prove that the equation

$$x^2+y^2+2gx+2fy+c+k(lx+my+n)=0$$

represents a circle through the points of intersection of

$$x^2+y^2+2gx+2fy+c=0 \quad \text{and} \quad lx+my+n=0$$

for all values of k.

The line $x=2$ cuts the circle $x^2+y^2=9$ in A, B; find the equation of the circle described on AB as diameter.

20. If $lx+my+n=0$ is a tangent to the circle

$$x^2+y^2+2gx+2fy+c=0,$$

then it is a tangent to the circle

$$x^2+y^2+2gx+2fy+c+k(lx+my+n)=0$$

for every value of k.

21. If $S\equiv x^2+y^2+2gx+2fy+c$ and $u\equiv lx+my+n$, interpret the equation $S+ku=0$ with respect to the circle $S=0$ and the line $u=0$.

If $u=0$ cuts $S=0$ in A and B, find the value of k when $S+ku=0$ represents the circle on AB as diamoter.

22. If A and B are conjugate points with respect to a circle, prove that the square on AB is equal to the sum of the squares on the tangents from A and B to the circle.

23. If A and B are conjugate points with respect to a circle, prove that the circle on AB as diameter cuts the given circle orthogonally.

24. Find the equation of the circle passing through $(1, 2)$ and orthogonal to the circles

$$x^2+y^2-5x+4=0, \quad x^2+y^2+8x+4=0.$$

25. Prove that every circle through the points $(b, 0)$, $(-b, 0)$ is orthogonal to all the circles of the system specified by the equation $x^2+y^2-kx+b^2=0$, where k is the parameter of the system.

26. Prove that $(b, 0)$ and $(-b, 0)$ are inverse points with respect to all the circles of the system $x^2+y^2-kx+b^2=0$, where k is the parameter.

27. Find the equation of the circle passing through the point $(-1, 2)$ and orthogonal to the circles

$$x^2+y^2-4=0 \quad \text{and} \quad x^2+y^2-3x-4=0.$$

28. Let $x^2+y^2-a_1x-b=0$ and $x^2+y^2-a_2x-b=0$ represent two circles. Show (1) that a circle can be described passing through (x_1, y_1) and orthogonal to the two given circles, and (2) that two circles can be described touching the line $lx+my+n=0$ and orthogonal to the two given circles. Find the equation of the circle in (1), and also the equations of the two circles in (2).

29. If (h', k') is the inverse of (h, k) with respect to the circle

$$(x-1)^2+(y-2)^2=5,$$

prove that $h'=\dfrac{h^2+k^2+3h-4k}{h^2+k^2-2h-4k+5}$; $k'=\dfrac{2h^2+2k^2-4h-3k}{h^2+k^2-2h-4k+5}$.

30. Prove that the points (h, k) and (h', k'), where

$$h'=a+\frac{r^2(h-a)}{(h-a)^2+(k-b)^2} \quad \text{and} \quad k'=b+\frac{r^2(k-b)}{(h-a)^2+(k-b)^2},$$

are always inverse to each other with respect to a fixed circle, and find its equation.

31. Find the length of the least chord of the circle

$$x^2+y^2+2gx+2fy+c=0$$

which passes through an internal point (x_1, y_1).

32. If $\alpha\equiv x-y-1$, $\beta\equiv x-y-2$, $\gamma\equiv x+y-3$, $\delta\equiv x+y-4$, prove that $\alpha\beta+\gamma\delta=0$ is a circle passing through the intersections of $\alpha=0$ and $\gamma=0$, $\alpha=0$ and $\delta=0$, $\beta=0$ and $\gamma=0$, $\beta=0$ and $\delta=0$.

33. If $\alpha\equiv 2x-y+3$, $\beta\equiv 5x+3y-20$, $\gamma\equiv x-3y+14$, $\delta\equiv x+4y+1$, prove that $\alpha\beta=\gamma\delta$ is the equation of the circle circumscribing the quadrilateral whose sides, taken in order, are $\alpha=0$, $\gamma=0$, $\beta=0$, $\delta=0$.

34. Prove that constants p, q can be so chosen that the equation

$$(a_2x+b_2y+c_2)(a_3x+b_3y+c_3)+p(a_3x+b_3y+c_3)(a_1x+b_1y+c_1)$$
$$+q(a_1x+b_1y+c_1)(a_2x+b_2y+c_2)=0$$

shall represent the circle circumscribing the triangle whose sides are

$$a_1x+b_1y+c_1=0, \quad a_2x+b_2y+c_2=0, \quad a_3x+b_3y+c_3=0.$$

Prove that

$$p(3x-2y-3)(x+2y)+q(x+2y)(2x+3y+3)$$
$$+r(2x+3y+3)(3x-2y-3)=0$$

is the equation of the circumcircle of the triangle whose sides are

$$2x+3y+3=0, \quad 3x-2y-3=0, \quad x+2y=0,$$

if $p=8$, $q=-1$, $r=-5$.

35. If $S=0$ and $S'=0$ be the equations of two circles, interpret the equation $S-kS'=0$, where k is a constant.

If a line meet $S=0$ in P and Q, $S'=0$ in P' and Q', and $S-kS'=0$ in R, show that $RP.RQ:RP'.RQ'$ is constant for all positions of the line.

36. The tangents from two fixed points to a variable circle are of given lengths; show that the circle passes through two fixed points.

37. The equation of the circle whose diameter is the line joining the points in which $lx+my=1$ cuts

$$a(x^2+y^2)+2gx+2fy+c=0$$

is $$a(l^2+m^2)(x^2+y^2)-2(al+flm-gm^2)x$$
$$-2(am+glm-fl^2)y+2a+2gl+2fm+c(l^2+m^2)=0.$$

38. Prove that the circumcircle of the triangle formed by the lines

$$bx+cy+a=0, \quad cx+ay+b=0, \quad ax+by+c=0$$

passes through the origin if

$$(b^2+c^2)(c^2+a^2)(a^2+b^2)=abc(b+c)(c+a)(a+b).$$

39. If $P(x_1, y_1)$ is a point within the circle

$$x^2+y^2+2gx+2fy+c=0,$$

and AB a chord of the circle passing through P such that $AP=2PB$, find the length of AB. If $AP/PB=m/n$, find the length of AB.

40. Through the point $P(1, 1)$ is drawn a line of gradient 1 to meet the circle $x^2+y^2-2x-4y=0$ in A and B; find the lengths of PA and PB, using the equation

$$(x-x_1)/\cos\theta=(y-y_1)/\sin\theta=r.$$

41. Through the point $A(x_1, y_1)$ is drawn a line to meet the circle $x^2+y^2=a^2$ in P and Q, and to meet the polar of A in R; prove that

$$1/AP+1/AQ=2/AR,$$

that is, prove that R is the harmonic conjugate of A with respect to the points P and Q in which any secant through A meets the circle. (§ 44, III.).

[Use the equation $(x-x_1)/\cos\theta=(y-y_1)/\sin\theta=r$.]

42. Through the point (3, 4) is drawn a chord of the circle

$$x^2+y^2=225,$$

so that the given point is a point of trisection of the chord; find the equation of the chord.

43. Through the point $A(1, 1)$ are drawn the two chords of the circle $x^2+y^2=10$ which are trisected at A; find the equations of the chords.

CHAPTER IX.

CONCHOID. CISSOID. WITCH. PARABOLA. ELLIPSE. HYPERBOLA.

64. The Conchoid of Nicomedes. Let O (Fig. 53) be a fixed point (called the pole) and AB a fixed straight line (called the directrix); let OPQ be a variable line cutting AB in Q and let the distance QP (measured either way) be constant. The locus of P is called the Conchoid of Nicomedes.

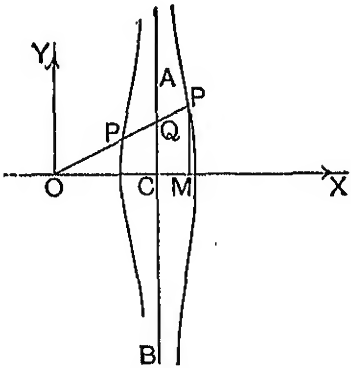


FIG. 53 (a).

Let $X'OX$ be drawn perpendicular to AB, meeting AB in C; let $X'OX$, $Y'OY$ be rectangular axes and let $OC=a$, $PQ=b$.

To find the equation of the conchoid.

Let (x, y) be the coordinates of any position of P, and let MP be the ordinate of P.

The triangles OCQ, OMP are similar; therefore

$$\frac{CQ}{OC}=\frac{MP}{OM} \text{ or } \frac{CQ}{c}=\frac{y}{x},$$

so that $CQ=\frac{cy}{x}$, and therefore the coordinates of Q are $\left(c, \frac{cy}{x}\right)$.

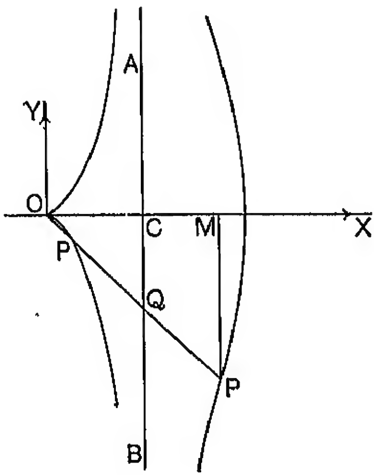


FIG. 53 (b).

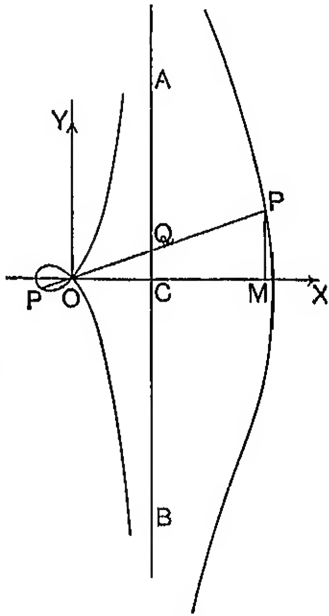


FIG. 53 (c).

The defining property of the conchoid is

$$PQ^2=b^2.$$

Hence $$(x-c)^2+\left(y-\frac{cy}{x}\right)^2=b^2,$$

which reduces to $$(x^2+y^2)(x-c)^2=b^2x^2.$$

This is the equation of the conchoid.

The curve has three forms according as (1) $b < c$ (Fig. 53 (*a*)), (2) $b = c$ (Fig. 53 (*b*)), (3) $b > c$ (Fig. 53 (*c*)). A point O, a line AB, and a length (or parameter) b being chosen, the locus may be roughly sketched by hand, as a circle may be roughly drawn by hand instead of with a pair of compasses.

The locus may be mechanically described, as a circle is described with a pair of compasses, with the instrument sketched in Fig. 54. The fixed point O is a pin projecting from a small wooden board; the variable line OPQ is a slot cut in a thin slip of wood, resting on the board so that the slip is movable about O in such a way that Q, a pin fixed on the under side of the slip, moves up and down AB, a straight groove cut in the board, while P, another fixed pin or pencil-point, traces out the conchoid on the board.

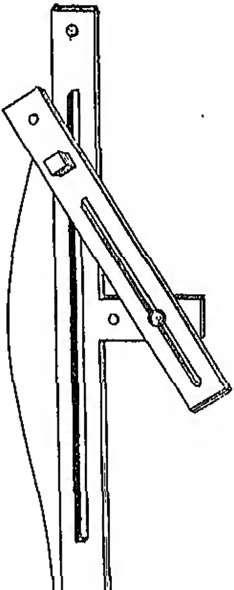


FIG. 54.

65. Geometrical Problems. The Conchoid of Nicomedes may be used to solve problems, just as the straight line and circle (ruler and compasses) are used. Just as we say, with centre O and radius r describe a circle, so we say with pole O, directrix AB and parameter b, describe a conchoid. Problems requiring the use of the straight line and circle only are called Euclidean problems; problems requiring the use of other loci, such as the conchoid, are called Geometrical Problems. Indeed, loci like the conchoid were invented to solve problems beyond the power of the straight line and circle, such as that of **trisecting an angle** and that of **finding two mean proportionals or duplicating the cube.**

66. Trisection of an Angle. Let ABC (Fig. 55) be a right-angled triangle having B a right angle. Describe a conchoid having A as pole, BC as directrix and $2AC$ as parameter.

Let a parallel to AB through C meet the conchoid in E. Then AE trisects the angle BAC.

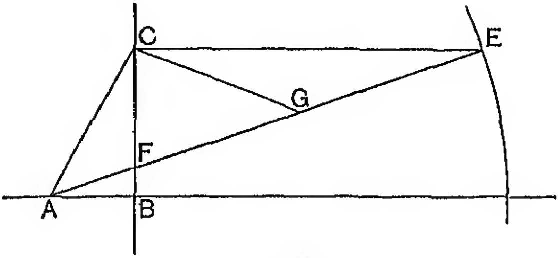


FIG. 55.

Proof. Let AE cut BC in F, and let G be the middle point of EF.

Then, since ECF is a right angle and $EG=GF$,

$$GC=GE=GF.$$

But

$$EF=\text{parameter of conchoid}$$
$$=2AC;$$

therefore $CA=CG=GE$,

and

$$C\hat{A}F=C\hat{G}F=2C\hat{E}F=2F\hat{A}B,$$

so that AE trisects angle BAC.

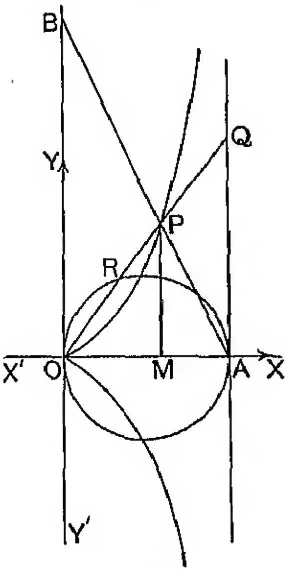


FIG. 56.

67. The Cissoid of Diocles. Let $X'OX$, $Y'OY$ (Fig. 56) be rectangular axes. Let A be a fixed point on the x-axis, and let $OA=a$. Describe the circle on OA as diameter; through A draw the perpendicular to OA. Let Q be a variable point on this perpendicular, join OQ cutting the circle in R and cut off $OP=RQ$. The locus of P is called the Cissoid of Diocles.

To find the equation of the Cissoid.

Let the coordinates of P be (x, y) and let MP be the ordinate of P.

Then $$\frac{AQ}{OA}=\frac{MP}{OM} \quad \text{or} \quad \frac{AQ}{a}=\frac{y}{x},$$

and therefore $$AQ=\frac{ay}{x}. \qquad \text{.........................(1)}$$

But $$OQ^2=OA^2+AQ^2=a^2+\frac{a^2y^2}{x^2}=\frac{a^2(x^2+y^2)}{x^2};$$

therefore $$OQ=\frac{a\sqrt{x^2+y^2}}{x}. \qquad \text{.................(2)}$$

Now $OQ \,.\, RQ=AQ^2$, and therefore $OQ \,.\, OP=AQ^2$.

Hence, by (1) and (2), since $OP=\sqrt{(x^2+y^2)}$,

$$\frac{a\sqrt{x^2+y^2}}{x} \cdot \sqrt{x^2+y^2}=\frac{a^2y^2}{x^2},$$

that is, $$x(x^2+y^2)=ay^2$$

or $$y^2=\frac{x^3}{a-x}.$$

This is the equation of the Cissoid. The locus may be roughly sketched by hand; its form is shown in the figure.

68. **The Duplication of the Cube.** Let d denote the edge of a cube; it is required to construct geometrically d_1 so that $d_1^3=2d^3$. This is the problem known as the duplication of the cube. In Fig. 56, let B be the point on OY such that $OB=2OA$, and let AB cut the Cissoid in P.

Then, from the equation

$$y^2=\frac{x^3}{a-x},$$

we have $$MP^2=\frac{OM^3}{OA-OM}=\frac{OM^3}{MA}. \qquad \text{..............(1)}$$

But, by similar $\triangle$s MAP, OAB,

$$\frac{MA}{MP}=\frac{OA}{OB}=\frac{1}{2},$$

and therefore $$MA=\tfrac{1}{2}MP.$$

Substitute in (1);

then $$MP^2 = \frac{OM^3}{\frac{1}{2}MP} \quad \text{or} \quad MP^3 = 2OM^3.$$

Now construct d_1 so that

$$OM : MP = d : d_1.$$

Then $$d_1^3 = 2d^3.$$

69. The Witch of Agnesi. Let $X'OX$, $Y'OY$ (Fig. 57) be rectangular axes, A a fixed point on $X'OX$, B the middle point of OA, CD the parallel through B to $Y'OY$. Let Q be a variable point on CD and let OQ meet the circle on OA as diameter in R. Let the parallel to OX through Q meet the parallel to OY through R in P. The locus of P is called the Witch of Agnesi.

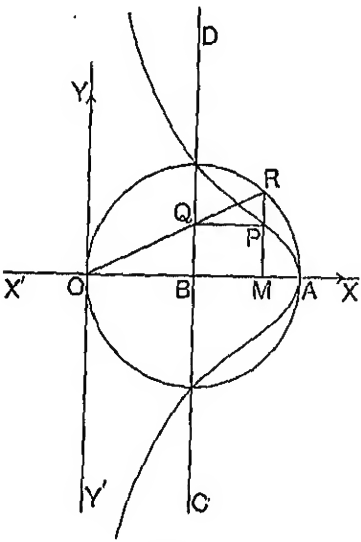


FIG. 57.

To find the equation of the Witch.

Let $OA = 2a$; let P be the point (x, y) and MP the ordinate of P.

From similar triangles OMR, ORA,

$$\frac{OR}{OA} = \frac{OM}{OR};$$

therefore $$OR^2 = OA \cdot OM = 2ax. \quad \text{.............(1)}$$

From similar triangles OBQ, OMR,

$$\frac{OQ}{OB} = \frac{OR}{OM};$$

therefore $$\frac{OQ^2}{OB^2} = \frac{OR^2}{OM^2},$$

that is, by (1), $$\frac{OQ^2}{a^2}=\frac{2ax}{x^2}$$

or $$OQ^2=\frac{2a^3}{x}. \quad\text{.............................}(2)$$

But $$OQ^2=OB^2+BQ^2=OB^2+MP^2=a^2+y^2;$$

therefore, by (2), $$a^2+y^2=\frac{2a^3}{x}$$

or $$x(a^2+y^2)=2a^3.$$

This is the equation of the Witch. The form of the curve is shown in the figure.

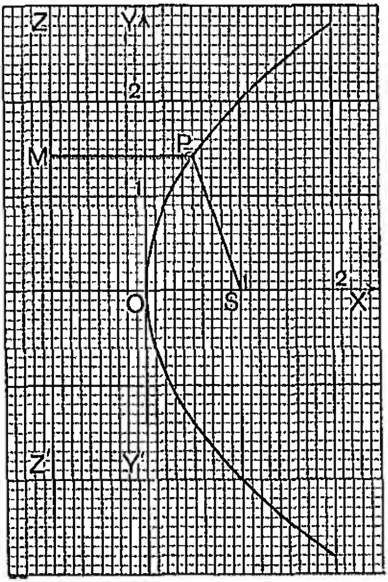


FIG. 58.

70. The Parabola. Let S (Fig. 58) be a fixed point, called the **focus**, $Z'Z$ a fixed straight line, called the **directrix**; let P be a variable point which moves so that its distance from S is numerically equal to its (perpendicular) distance

from $Z'Z$; the locus of P is called a **Parabola.** The form of the curve is shown in Fig. 58 when S is the point (1, 0) and $Z'Z$ the line $x=-1$ referred to the axes $X'OX$, $Y'OY$. It was obtained by describing circles with S as centre and radii 10, 11, 12, 13 ... divisions in length to cut successively the vertical lines from O to the right; *e.g.* the point P on the curve is such that radius $SP=15$ divisions in length $=PM$, the perpendicular from P to the directrix. The point O is called the **vertex** and the line OX the **axis** of the parabola.

To find the equation of the parabola in Fig. 58. Let P (x, y) be any point on the curve.

Then $$SP^2=PM^2;$$

therefore $$(x-1)^2+y^2=(x+1)^2;$$

that is, $$y^2=4x$$

is the equation of the parabola.

Ex. 1. Find the equation of the parabola whose focus is the point $(a, 0)$ and whose directrix is the line $x=-a$.

Ex. 2. Find the equation of the parabola whose focus is the point (2, 0) and whose directrix is the y-axis.

Ex. 3. Find the equation of the parabola whose focus is the point $(0, a)$ and whose directrix is the line $y=-a$.

Ex. 4. Find the equation of the parabola whose focus is the point (2, 1) and whose directrix is $3x+4y=5$.

71. The Ellipse. Let S and S' be two fixed points in a plane (called the **foci**) and let a variable point P in the plane move so that $PS+PS'$ is constant; then the locus of P is called an **ellipse** with foci S, S'. The locus may be mechanically described by passing an endless string round two pins, placed at the foci S, S', and then keeping the string tight by a pencil moving in the plane and tracing out the locus.

Fig. 59 shows the form of the ellipse when the foci S, S' are the points (2, 0), (−2, 0) and $PS+PS'=5$.

To find the equation of this ellipse.

Let $P(x, y)$ be any point on the locus; then

$$SP=\sqrt{(x-2)^2+y^2}, \quad S'P=\sqrt{(x+2)^2+y^2};$$

therefore $$\sqrt{(x+2)^2+y^2}+\sqrt{(x-2)^2+y^2}=5;$$

or $$\sqrt{(x+2)^2+y^2}=5-\sqrt{(x-2)^2+y^2}.$$

Squaring, we obtain

$$(x+2)^2+y^2=25+(x-2)^2+y^2-10\sqrt{(x-2)^2+y^2},$$

which reduces to

$$8x-25=-10\sqrt{(x-2)^2+y^2}.$$

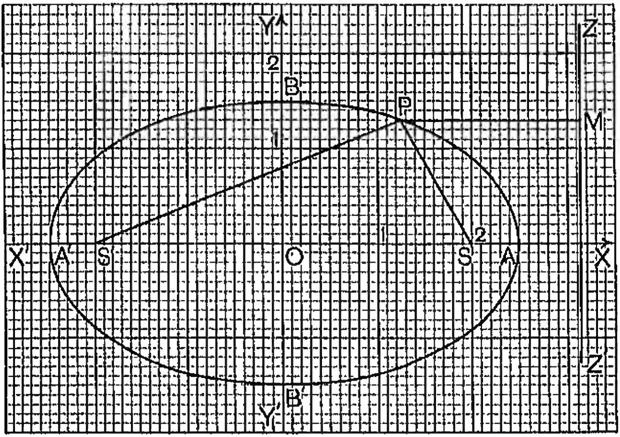


FIG. 59.

Squaring again, we get

$$64x^2-400x+625=100x^2-400x+400+100y^2,$$

or, $$36x^2+100y^2=225.$$

This is the equation of the ellipse.

AA' is called the major axis and BB' the minor axis of the ellipse; the points A and A' are called the vertices of the ellipse.

Ex. 1. A point P moves so that the sum of its distances from the points (2, 0) and (−2, 0) is 6; find the equation of the ellipse traced out by P; and draw the figure.

Ex. 2. A point P moves so that the sum of its distances from the points $(c, 0)$ and $(-c, 0)$ is $2a$, where $a > c$; prove that the equation of the ellipse traced out is

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1 \quad \text{or} \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where $b^2=a^2-c^2$.

Ex. 3. A line MN, 7 inches long, slides with the end M on the x-axis and the end N on the y-axis; P is the point on MN, that is 3 inches from M and 4 inches from N. If x, y are the coordinates of P, show that

$$x=\frac{4}{7}OM, \quad y=\frac{3}{7}ON, \quad \frac{x^2}{16}+\frac{y^2}{9}=1.$$

If P is b inches from M and a inches from N, the length of MN being now $(a+b)$ inches, then

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

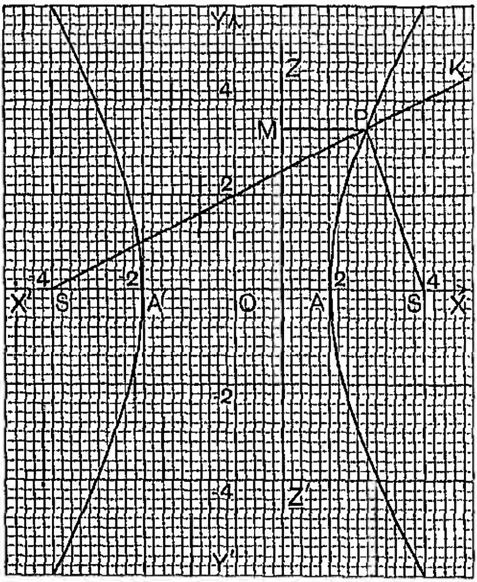


FIG. 60.

72. The Hyperbola. Let S and S' be two fixed points in a plane (called the foci) and let a variable point P in the plane move so that the difference of PS and PS' is constant;

then the locus of P is called a hyperbola with foci S, S'. The locus may be mechanically described as shown in Fig. 60. The rod $S'K$ turns about S', while a string (whose length is less than that of the rod) connected to S and K is kept tight by a pencil P moving in the plane along the rod.

If S is the point $(4, 0)$, S' the point $(-4, 0)$ and $PS' \sim PS = 4$, to find the equation of the locus.

Let $P(x, y)$ be any point on the locus (Fig. 60).

$$SP = \sqrt{(x-4)^2 + y^2}, \quad S'P = \sqrt{(x+4)^2 + y^2}.$$

If
$$S'P - SP = 4, \quad \text{.......................}(1)$$

$$\sqrt{(x+4)^2 + y^2} - \sqrt{(x-4)^2 + y^2} = 4;$$

therefore
$$\sqrt{(x+4)^2 + y^2} = 4 + \sqrt{(x-4)^2 + y^2}.$$

Squaring and reducing, we have

$$2x - 2 = \sqrt{(x-4)^2 + y^2}.$$

Squaring again, we get

$$4x^2 - 8x + 4 = (x-4)^2 + y^2$$

or
$$3x^2 - y^2 = 12,$$

which is the equation of the hyperbola.

The same equation is obtained if we start from

$$SP - S'P = 4 \quad \text{.......................}(2)$$

instead of from (1). The right-hand branch of the curve corresponds to (1), and the left-hand branch to (2).

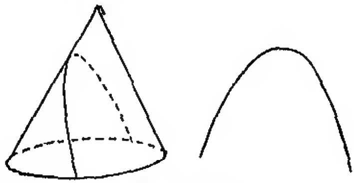
73. Conic Sections. If a right circular cone be cut by a plane

(i) which is parallel to a generator, the section is a *parabola*;

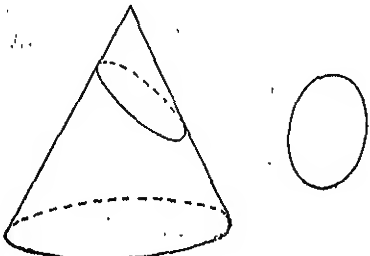
(ii) which is not parallel to a generator and yet cuts only one sheet of the complete conical surface, the section is a *circle* when the plane is perpendicular to the axis of the cone, and an *ellipse* when it is not;

(iii) which cuts both sheets of the conical surface, the section is a *hyperbola*.

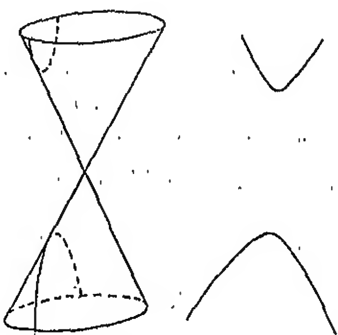
Fig. 61 helps to explain these statements. The parabol ellipse and hyperbola are often referred to as conic section



(a) The Parabola.



(b) The Ellipse.



(c) The Hyperbola.

FIG. 61.

The definitions of parabola, ellipse and hyperbola gi in §§ 70, 71, 72 do not show the connection between

curves as clearly as might be; hence the following definition, called the focus and directrix definition, is also worth noting:

If S is a fixed point, called the *focus*, and $Z'Z$ a fixed line called the *directrix*, and if M is the projection on $Z'Z$ of a variable point P which moves so that $SP = e \, . \, PM$, then the locus of P is called a parabola if $e=1$, an ellipse if $e<1$ and a hyperbola if $e>1$, e being called the *eccentricity*.

Ex. 1. Find the equation of the conic section whose focus is the point (2, 0), whose directrix is $x=\frac{25}{8}$ and whose eccentricity is $\frac{4}{5}$.

The equation is $\qquad (x-2)^2+y^2=\frac{16}{25}(x-\frac{25}{8})^2,$

that is, $$36x^2+100y^2=225.$$

The conic is the ellipse of § 71.

Ex. 2. Find the equation of the conic section whose focus is the point (4, 0), whose directrix is $x=1$ and whose eccentricity is 2.

The equation is $\qquad (x-4)^2+y^2=4(x-1)^2,$

that is, $$3x^2-y^2=12.$$

The conic is the hyperbola of § 72.

Ex. 3. Find the equations of the conic sections whose focus is the point (2, 1), whose directrix is $x-2y+3=0$ and whose eccentricities are (i) $\frac{1}{2}$; (ii) 1; (iii) 2.

(i) The equation is

$$(x-2)^2+(y-1)^2=\frac{1}{4}\cdot\left(\frac{x-2y+3}{\sqrt{5}}\right)^2,$$

that is, $$19x^2+4xy+16y^2-86x-28y+91=0.$$

(ii) The equation is

$$(x-2)^2+(y-1)^2=\left(\frac{x-2y+3}{\sqrt{5}}\right)^2,$$

that is, $$4x^2+4xy+y^2-20x+2y+16=0.$$

(iii) The equation is

$$(x-2)^2+(y-1)^2=4\left(\frac{x-2y+3}{\sqrt{5}}\right)^2,$$

that is, $$x^2+16xy-11y^2-44x+38y-11=0.$$

Ex. 4. Show that the general equation of a conic section is of the second degree in x, y.

Let the focus be (p, q), the directrix $lx+my+n=0$ and the eccentricity e; then the equation of the conic is

$$(x-p)^2+(y-q)^2=e^2\frac{(lx+my+n)^2}{l^2+m^2}.$$

Squaring out, collecting like terms and rearranging, we get an equation which contains terms in x^2, xy, y^2, x, y and an absolute term; the equation is therefore of the form

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

74. The Equation of a Locus. A geometrical locus such as the straight line, circle, conchoid, cissoid, etc., is defined by a certain condition. A fundamental problem in Analytical Geometry is to represent a defined geometrical locus by an analytical equation. This can be done in an infinite number of ways; for rectangular axes of reference can be chosen in an infinite number of ways. If (x, y) be the coordinates, with respect to chosen or assigned axes, of *any* point on a locus, the condition defining the locus can be translated into an equation in x, y, and certain constants required to specify the locus. This equation is called the equation of the locus.

75. Worked Examples. We shall now work some examples of the process of finding the analytical equations of specified loci.

Ex. 1. *If O is the origin of rectangular axes and Q moves round the circle $x^2+y^2-4x+3=0$, find the equation of the locus of P, the middle point of OQ. Draw the loci of Q and P.*

Let (h, k) be the coordinates of a position of P (Fig. 62).

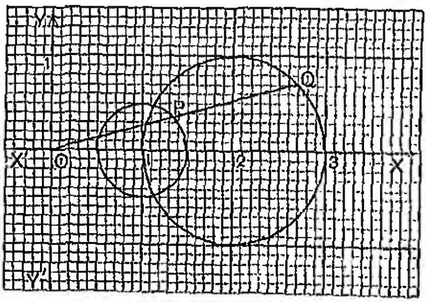


FIG. 62.

Then $(2h, 2k)$ are the coordinates of the corresponding position of Q.

But Q is a point on the given circle; therefore the coordinates of Q satisfy the equation

$$x^2+y^2-4x+3=0; \quad \text{.................................(i)}$$

$$\therefore\ (2h)^2+(2k)^2-4(2h)+3=0;$$

$$\therefore\ 4h^2+4k^2-8h+3=0;$$

i.e. the coordinates of *any* point on the locus of P satisfy the equation
$$4x^2+4y^2-8x+3=0, \qquad \text{(ii)}$$
obtained by writing x, y for h, k to indicate a variable point.

Hence this is the equation of the locus of P.

We may write (i) in the form
$$(x-2)^2+y^2=1,$$
and (ii) in the form
$$(x-1)^2+y^2=(\tfrac{1}{2})^2.$$

Hence the locus of Q is the circle centre (2, 0), radius 1; and the locus of P is the circle centre (1, 0), radius $\frac{1}{2}$.

Ex. 2. *A variable circle touches the x-axis and the fixed circle whose radius is a, and centre* (0, a); *find the equation of the locus of the centre of the variable circle, and sketch the form of the locus.*

Let A (Fig. 63) be the centre of the fixed circle.

Let $P(h, k)$ be a position of the centre of the variable circle; let MP be the ordinate of P.

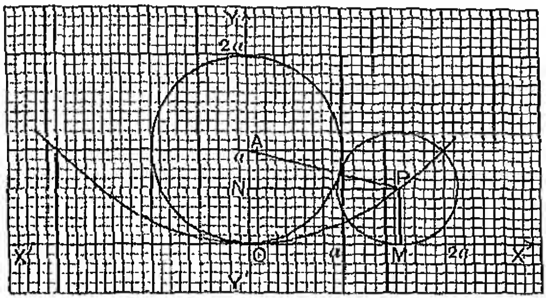


FIG. 63.

Draw PN, the perpendicular from P to OA.

Then
$$AP^2=NA^2+NP^2. \qquad \text{(i)}$$

Also
$$AP = \text{sum of the radii of the two circles}$$
$$= a+MP=a+k;$$
$$NA=OA-ON=a-k;$$
and
$$NP=h.$$

Substituting in (i), we get
$$(a+k)^2=(a-k)^2+h^2,$$
which reduces to
$$h^2=4ak.$$

Writing x, y for h, k to denote a variable point on the locus, we get
$$x^2=4ay$$
as the equation of the locus, the form of which is shown in the figure. The locus is a parabola of which O is the vertex and OY the axis (§ 70).

Ex. 3. *A variable line passing through the point* (1, 1) *meets the axes of x and y at M and N respectively. Parallels through M and N, to the axes of y and x respectively, meet in P. Find the equation of the locus of P, and draw the form of the locus.*

Let one position of the variable line through $A(1, 1)$ be MAN of gradient m (Fig. 64), and let $P(h, k)$ be the corresponding point on the locus.

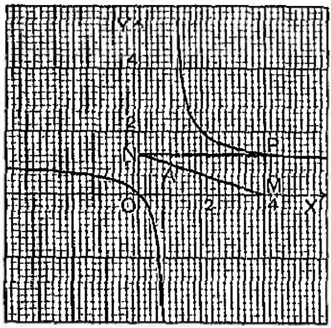


FIG. 64.

Then the equation of MAN is

$$y-1=m(x-1). \quad \text{(i)}$$

Now M, whose coordinates are $(h, 0)$, lies on the line (i);

therefore $$-1=m(h-1). \quad \text{(ii)}$$

Also N, whose coordinates are $(0, k)$, lies on (i);

therefore $$k-1=-m. \quad \text{(iii)}$$

We wish to obtain a relation between h, k, so divide (ii) by (iii), and get

$$-\frac{1}{k-1}=-h+1,$$

that is, $$1=(h-1)(k-1)$$

or $$hk=h+k.$$

Writing x, y for h, k to denote a variable point on the locus, we find

$$xy=x+y \quad \text{(iv)}$$

as the equation of the locus, whose form is shown in Fig. 64.

If we write equations (ii) and (iii) in the form

$$h=1-1/m, \quad k=1-m,$$

we see that $x=1-1/m$, $y=1-m$ are freedom equations of the locus. Equation (iv) is the constraint equation, obtained of course by the elimination of m. The locus is a hyperbola (§ 72).

Ex. 4. *A and A′ are the points $(a, 0)$ and $(-a, 0)$; and B and B′ are the points $(0, b)$ and $(0, -b)$. If Q and Q′, variable points in A′A, divide it externally and internally in the same ratio, and if BQ and B′Q′ meet in P, find the equation of the locus of P, and sketch the locus.*

Let Q divide $A'A$ externally in the ratio $k:1$ (Fig. 65).

Then
$$\text{abscissa of } Q = \frac{ka+a}{k-1};$$

therefore
$$Q \text{ is the point } \left(\frac{a(k+1)}{k-1},\ 0\right).$$

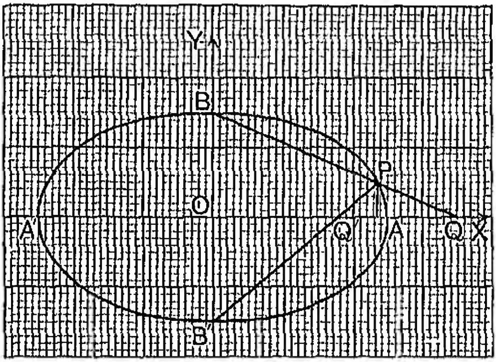


FIG. 65.

Also Q' divides $A'A$ internally in the ratio $k:1$;

herefore
$$\text{abscissa of } Q' = \frac{ka-a}{k+1},$$

nd
$$Q' \text{ is the point } \left(\frac{a(k-1)}{k+1},\ 0\right).$$

The equation of BQ is
$$\frac{x(k-1)}{a(k+1)} + \frac{y}{b} = 1. \qquad\text{(i)}$$

The equation of $B'Q'$ is
$$\frac{x(k+1)}{a(k-1)} - \frac{y}{b} = 1. \qquad\text{(ii)}$$

If then (p, q) is a point P on the locus, we have
$$\frac{p(k-1)}{a(k+1)} = 1 - \frac{q}{b}, \quad \text{from (i)},$$

id
$$\frac{p(k+1)}{a(k-1)} = 1 + \frac{q}{b}, \quad \text{from (ii)}.$$

Multiplying these equations together so as to eliminate k, we have

$$\frac{p^2}{a^2}=1-\frac{q^2}{b^2}.$$

This is an equation connecting p, q with the constants specifying the locus. Write x, y for p, q, and we obtain

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

as the equation of the locus. A sketch of the locus is shown where $a=5$, $b=3$; the value of k for the points Q, Q' in the figure is 6. The locus is an ellipse (§ 71).

Ex. 5. *Q and R are variable points on the x and y axes, such that QR subtends a right angle at the fixed point $A\,(a, b)$; and P is the foot of the perpendicular from the origin to QR. Find the equation of the locus of P.*

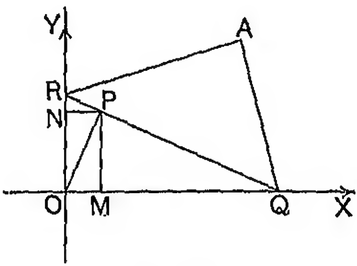


FIG. 66.

Let $P(h, k)$ (Fig. 66) be a point on the locus, and let M, N be the projections of P on the x and y axes.

Let $OQ=t$;

then $\qquad$ gradient of $AQ=\dfrac{b}{a-t}$;

therefore $\qquad$ gradient of $AR=\dfrac{t-a}{b}$.

Hence the equation of AR is

$$y-b=\frac{t-a}{b}(x-a). \qquad \text{(i)}$$

But R lies on AR; substituting $x=0$ and $y=OR$ in (i), we get

$$OR-b=-\frac{a(t-a)}{b}$$

or

$$b\,.\,OR=a^2+b^2-at. \qquad \text{(ii)}$$

Now, from the right-angled triangle OPQ, we get

$$OQ \cdot OM = OP^2$$

that is
$$t = \frac{h^2+k^2}{h}. \quad \text{.........................(iii)}$$

From the right-angled triangle OPR, we get

$$OR \cdot ON = OP^2$$

or
$$OR = \frac{h^2+k^2}{k}. \quad \text{.........................(iv)}$$

Substituting in (ii) the values of t and OR from (iii) and (iv), we get

$$\frac{b(h^2+k^2)}{k} = a^2+b^2 - \frac{a(h^2+k^2)}{h},$$

that is,
$$(h^2+k^2)(bh+ak) = (a^2+b^2)hk. \quad \text{.......................(v)}$$

Writing x, y for h, k to denote a variable point on the locus, we get

$$(x^2+y^2)(bx+ay) = (a^2+b^2)xy$$

as the equation of the locus of P.

Note. From (iv), $OR = (h^2+k^2)/k$, so that (ii) may be written

$$b(h^2+k^2)/k = a^2+b^2 - at. \quad \text{......................(iia)}$$

Equation (v) is found by eliminating t from (ii$a$) and (iii).

The method of solution thus consists in first choosing a suitable parameter t, then forming two equations in h, k, t, and finally eliminating t. The last two steps again illustrate the connection between freedom and constraint equations.

EXERCISES XIX.

1. If A be the fixed point $(0, 2a)$, and Q a variable point which moves along the x-axis, find the equation of the locus of the middle point of AQ, and draw the locus.

2. P is a variable point lying within the angle XOY; M and N are the projections of P on OX and OY respectively. If the perimeter of the rectangle $OMPN$ is 4, find the equation of the locus of P, and draw the locus.

3. If in Ex. 2 the area of $OMPN$ is 1, find the equation of the locus of P, and sketch the locus.

4. $OABC$ is a variable rectangle of constant perimeter $2a$, and the sides OA and OC lie along the axes of reference; find the equation of the locus of the middle point of AC, and draw the locus when $a=1$.

5. A is the fixed point $(1, 1)$ and AB is any line through it cutting the x-axis in B. If AC is perpendicular to AB and meets the y-axis in C, find the equation of the locus of P, the middle point of BC, as AB varies.

6. A and B are any two points on the axes of x and y respectively such that $2OA+3OB=10$; find the equation of the locus of the middle point of AB, and draw the locus.

7. Q is a variable point on the circle $x^2+y^2=a^2$; QP is drawn parallel to the x-axis so that $QP=2$. Find the equation of the locus of P; interpret the equation and draw the locus.

8. Q is a variable point on the circle $x^2+y^2=9$; QM and MP are drawn parallel to the x- and y-axes respectively so that $QM=2$, $MP=3$; find the equation of the locus of P, and draw the locus.

9. MQ is a variable ordinate of the circle $x^2+y^2=1$; P is taken in MQ so that $MP=2MQ$. Find the equation of the locus of P, and sketch the locus. Find also the area enclosed by the locus.

10. MQ is a variable ordinate of the circle $x^2+y^2=a^2$; QR, RP are drawn parallel to the y- and x-axes respectively, so that $MR=2MQ$ and $RP=OM$, where O is the origin; find the equation of the locus of P, and sketch the locus.

11. If O is the origin and Q moves round the circle

$$x^2+y^2-4x+3=0,$$

find the equation of the locus of P, the point of trisection of OQ nearest to O. Draw the locus.

12. A is the fixed point $(b, 0)$; APQ is a variable secant of the fixed circle $x^2+y^2=a^2$; find the equation of the locus of the middle point of the chord PQ, and draw the locus.

13. A variable point P moves so that its distance from the x-axis is numerically equal to its distance from the point $(0, 2a)$; find the equation of the locus of P, and sketch the locus.

14. A variable point P moves so that its distance from the point $(0, 4)$ is numerically equal to its distance from the line $y=1$; find the equation of the locus of P, and sketch the locus.

15. A variable point P moves so that its distance from the point $(8, 0)$ is double its distance from the y-axis; find the equation of the locus of P, and sketch the locus.

16. A variable straight line cuts $X'OX$, $Y'OY$ in P, Q respectively, and moves so that the area OPQ is constant $(=a^2)$; find the equation of the locus of the middle point of PQ, and sketch the locus.

17. A straight line PQ of constant length $2a$ slides between the axes of x and y; find the equation of the locus (i) of the middle point of PQ, (ii) of each of the points of trisection of PQ. Sketch the forms of the loci.

18. Prove that $y=x-x^2$ is the locus of a point which moves so that its distance from the point $(1/2, 0)$ is always equal to its distance from $y=1/2$. Sketch the locus.

Make a drawing of the ellipse whose focus is at the origin, e directrix is $x-y=3$ and whose eccentricity is 1/2. Find the ion of the curve.

A straight line rotates in a plane about a fixed point A, whose inates with respect to rectangular axes OX and OY in the plane $=a$ and $y=b$, and cuts the axes in the variable points Q and R. int P is taken on the line so that $PQ=RA$. Show that the ion of the locus of P is the hyperbola $xy=ab$, and sketch the

B is a fixed point on the y-axis such that $OB=k$. C is any on the x-axis, OD bisects the angle BOC and meets BC in D, is the middle point of CD. Find the equation of the locus is C moves from O to a point A along OX. Draw the path on d paper, taking k as 5 cms. and OA as 30 cms.

A variable circle touches the x-axis and the fixed circle whose is $(0, a)$ and radius a; find the equation of the locus of the on the variable circle which is furthest from (i) the x-axis, y-axis; and sketch the forms of the loci.

A fixed circle, centre $(0, b)$ and radius a, is drawn. A variable ouches the fixed circle and the axis of x. Find the equation of us of the centre of the variable circle (i) when $b>a$, (ii) when ii) when $b<a$.

A variable circle is described to pass through the point $(a, 0)$ touch the straight line $y=x$. Find the equation of the locus ontre of the variable circle, and sketch the locus.

variable circle is described to pass through the point $(0, a)$ ch the straight line $y=x$. Find the equation of the locus of e, and sketch the locus.

variable circle is described to pass through the point $(a, 0)$ ch the line $x+y=0$; find the equation of the locus of its and sketch the locus.

variable circle passes through the point (a, a) and touches is. Find the equation of the locus of its centre, and sketch s.

fixed circle of radius a touches the x-axis at the origin. A circle touches the y-axis and the fixed circle; find the of the locus of the centre of the variable circle, and sketch

nd the equations of the loci of the centres of the circles uch both the x-axis and the fixed circle $x^2+y^2=a^2$. Sketch and refer each sketch to its corresponding equation.

variable circle touches OX and the line $x=a$. The join igin to the centre of the circle meets the circle in P. Find ion of the locus of P, and sketch the locus.

31. AOA', BOB' are two perpendicular diameters of a circle whose centre is O, and whose radius is unity. R is a movable point on the circle, $A'R$ meets BOB' in N; and Q is a point on AR whose distance from BOB' is equal to ON. Find the equation of the locus of Q, $A'OA$ and $B'OB$ being the x- and y-axes of reference. Trace the locus on squared paper, taking special care to show the form near the point whose abscissa is unity.

32. A is the fixed point $(a, 0)$; Q is a variable point on the y-axis, and AQP a variable isosceles triangle on AQ as base, having QP parallel to the line $y=x$. Prove that the equation of the locus of P is

$$x^2-y^2+2ax=a^2.$$

Trace the curve.

33. P is the foot of the perpendicular from the origin on to a movable line cutting the axes at A and B so that $OA+OB=1$. Prove that the locus of P is specified by the equation

$$(x^2+y^2)(x+y)=xy.$$

From considerations of its geometrical property, sketch roughly the part of the curve that lies within the angle XOY.

34. $OABC$ is a square; D is a fixed point on OA produced. A variable line DPQ meets AB in P and BC in Q; prove that the locus of the intersection of OP and AQ is a straight line.

35. A circle, described with the origin O as centre and radius a, meets the negative part of the axis of x in A. P is any point on this circle, and Q is a point on the ordinate of P such that $OQ=AP$. Prove that the locus of Q is a circle, centre $(a, 0)$ and radius $a\sqrt{3}$.

36. P is the foot of the perpendicular from the origin to a tangent through the movable point Q on the circle on OA as diameter, where O is the origin and A is the point $(2a, 0)$. Prove that the equation of the locus of P is

$$(x^2+y^2)^2-2ax(x^2+y^2)-a^2y^2=0.$$

The locus is called the pedal of the circle with respect to the point O on it.

37. A circle is described on OA as diameter, where O is the origin and A is the point $(2a, 0)$. Q is any point on the circle, R is the image of Q in OA, and the diameter through R meets OQ in P. Prove that the locus of P is given by the equation

$$3x^2-y^2-2ax=0.$$

38. A is the point $(a, 0)$; B and C are variable points on the y axis such that $BC=a$. Prove that the locus of the foot of the perpendicular from C to AB is given by

$$x(x^2+y^2)-ax(2x+y)+a^2(x+y)=0.$$

39. A and B are the points $(a, 0)$ and $(0, b)$ respectively. Q is a movable point in the line AB, and M and N are its projections on

the axes of x and y respectively. Prove that the equation of the locus of the intersection of AN and BM is

$$b^2x^2+abxy+a^2y^2-2ab^2x-2a^2by+a^2b^2=0.$$

Sketch the locus.

40. A and B are the points $(a, 0)$ and $(0, b)$ respectively, and $OACB$ is a rectangle. Through C is drawn a variable line to meet the axes of x and y in Q and R respectively. BQ and AR meet in P; prove that the equation of the locus of P is

$$b^2x^2+abxy+a^2y^2=ab(bx+ay).$$

41. C is the fixed point (a, b); A and B are its projections on the x- and y-axes. Q in OA and R in BC are such that QR is parallel to OB; S in OB and T in AC are such that ST is parallel to OA. If QR and ST are movable, prove that the locus of the intersection of QS and RT is the line AB, and that the locus of the intersection of SR and QT is the line OC.

42. A is the point $(2a, 0)$; Q is a variable point on the circle on OA as diameter, where O is the origin. On the line OQ is measured, either way, a length QP equal to $2a$. Prove that the locus of Q (called a cardioid) is specified by the equation

$$(x^2+y^2-2ax)^2=4a^2(x^2+y^2).$$

43. A is the point $(a, 0)$ and B any point on the line $x=a$; the bisector of the angle OBA cuts OA at N, and from N a perpendicular is drawn to OB, meeting it at P. Find the equation to the locus of P as B moves along the line $x=a$.

If PN is produced to meet the line $x=a$ at Q, find the equation of the locus of the middle point of PQ, and show that the locus is a cissoid.

MISCELLANEOUS EXAMPLES I.

1. Prove that the points $(3, 4)$ and $(-4, 3)$ are equidistant from the origin.

2. Prove that the points $(\sqrt{3}, 1/2)$, $(7\sqrt{3}/2, 3)$, $(\sqrt{3}, 11/2)$ are the vertices of an equilateral triangle.

3. A, B, two points on an axis, have abscissae $(a+b)$, $(a-b)$ respectively. C and D are points on the axis such that

$$AC : CB = a : b = -AD : DB;$$

prove that $CD=4ab^2/(b^2-a^2)$.

4. A, B, C are the three points $(1, 4)$, $(3, 2)$, $(3, 11/2)$ respectively. M is the middle point of AB, and AC is produced its own length to N; calculate MN and the intercepts made by MN on the axes.

5. Prove that the points $(-3, 1)$, $(-1, 6)$, $(-5, -4)$ are collinear and find the ratio in which the first cuts the join of the second and third.

6. Prove that the lines joining $(-2, -3)$, $(6, 5)$ and $(1, -5)$, $(3, 7)$ are the diagonals of a parallelogram.

7. If (a, b), (c, d) are opposite vertices of a parallelogram and (c, b) is a third vertex, find the coordinates of the fourth vertex.

8. Find the coordinates of the intersection of the medians of the triangle whose vertices are $(5, -1)$, $(-3, -4)$, $(1, 8)$.

9. Find the coordinates of the centroid of the triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3).

10. Prove that the lines joining $(4, 0)$, $(-2, 3)$ and $(-3, 2)$, $(6, 2)$ trisect one another.

11. If $(-3, 2)$, $(1, 1)$, $(5, 7)$ are the middle points of the sides of a triangle, find the coordinates of the vertices of the triangle.

12. If masses 1, 1, 2 are placed at the points $(2, 6)$, $(4, -10)$, $(-1, 4)$, find the centroid of the masses.

13. If masses m_1, m_2, m_3 are placed at the points (x_1, y_1), (x_2, y_2), (x_3, y_3), find the centroid of the masses.

14. If masses m_1, m_2, ..., m_n are placed at the points (x_1, y_1), (x_2, y_2), ..., (x_n, y_n), find the centroid of the masses.

15. If G is the centroid of any number of fixed points A, B, C, etc., and P is a variable point, prove that

$$\Sigma PA^2 = \Sigma GA^2 + n \,.\, GP^2,$$

where n is the number of points.

16. If G is the centroid of masses m_1, m_2, etc., placed at the fixed points A_1, A_2, etc., and P is a variable point, prove that

$$\Sigma(m \,.\, PA^2) = \Sigma(m \,.\, GA^2) + (\Sigma m) \,.\, GP^2.$$

17. A particle starts from the point $(2, 3)$ and moves with component velocities of 3 and 4 feet per second parallel to the axes $X'OX$, $Y'OY$ respectively; prove that the position of the particle at time t seconds is specified by the equations

$$x = 2 + 3t, \quad y = 3 + 4t,$$

the scale unit of each axis being 1 foot.

Graph the line of motion and find its equation in the form

$$Ax + By + C = 0.$$

18. A particle starts from the point $(-4, -1)$ and one second later arrives at the point $(-2, 3)$; find freedom equations for its path and deduce the constraint equation.

19. The scale unit of each of the axes $X'OX$, $Y'OY$ is one foot. The motion of a particle in the plane of the axes is given by the equations
$$x=-1+2t, \quad y=2-t,$$
t being measured in seconds. Graph to a suitable scale the positions of the particle when t is $-3, -1, 0, 2, 3, 4$. What are the x- and y-components of the velocity of the particle, and what is the constraint equation of its path?

20. Draw two rectangular axes $t'Ot$, $v'Ov$. Let one inch, the scale unit on the t-axis, represent one second; let one inch, the scale unit on the v-axis, represent a velocity of 32 feet per second. Draw the straight line joining the origin to (1, 1). The diagram is called a **Velocity-Time** or **v-t Diagram** of the motion of a point on an axis. Find from the diagram

(1) the velocity when t is 0, 1, 2, 3, 4;

(2) at what times the velocity is (*a*) 16 ft. per sec., (*b*) 32 ft. per sec., (*c*) 48 ft. per sec., (*d*) 76·8 ft. per sec.;

(3) the acceleration (increase of velocity per second);

(4) the space described in 1 sec., in 3 secs., in the 3rd sec.

21. Taking the velocity-time diagram of Ex. 20, find general formulae specifying (1) the velocity v at time t, (2) the time t at which the velocity is v, (3) the space described in t secs., (4) the velocity v when the space described is s.

22. The velocity-time diagram of the motion of a point on an axis is a straight line. Show that the gradient of the line measures the acceleration of the motion.

23. If $v=u+\alpha t$ is the equation of the velocity-time diagram, what is the measure of (1) the acceleration of the motion, (2) the initial velocity, (3) the time when the particle is at rest?

24. Find freedom equations for the motion of a point along a straight line when the point has at one time coordinates (a, b) and t_1 seconds later coordinates (c, d). Deduce the constraint equation of the line.

25. Find freedom equations for the locus of a point which moves so that the gradient of the line joining it to (2, 1) is constantly 3/4.

26. Prove that the lines joining $(-4, 3)$, $(-2, 1)$ and $(-5, -1)$, $(-3, -3)$ are both perpendicular to the line joining $(5, -2)$, $(3, -4)$.

27. Find the equation of the perpendicular to $4x+y-2=0$ through the point $(-2, 3)$, the coordinates of the point of intersection, and the length of the perpendicular.

28. Prove that the coordinates of the foot of the perpendicular from the point (x_1, y_1) on the line $ax+by+c=0$ are
$$\frac{b(bx_1-ay_1)-ac}{a^2+b^2}, \quad \frac{a(ay_1-bx_1)-bc}{a^2+b^2}.$$

29. A point initially at (7, 2) moves so that its distances from the lines $3x-4y+1=0$, $8x+6y-3=0$ are in a constant ratio. Find the equation of the locus of the point and the value of the constant ratio.

30. Find the coordinates of the point which is equidistant from (2, 3), (5, 4), (3, −2).

31. Find the coordinates of the orthocentre of the triangle whose vertices are (2, 3), (5, 4), (3, −2).

32. If O, G, H are the circumcentre, centroid and orthocentre of the triangle whose vertices are (2, 3), (5, 4), (3, −2), calculate OG and GH.

33. Find the coordinates of the vertices of the squares described on the join of (3, 4) to the origin.

34. A, B are the points $(-a, b)$, $(c, -d)$ respectively. Through A is drawn AC equal and perpendicular to AB; find the coordinates of the two possible positions of C.

35. ABC is a triangle having C a right angle. On AB is described the square external to the triangle. If CA, CB are taken as axes of x and y, find the coordinates of the vertices of the square other than A, B in terms of a, b when $a=CB$, $b=CA$.

36. ABC is a triangle, right-angled at A; on BC, CA, AB are described, external to the triangle, squares $BCDE$, $CAFG$, $ABHK$, specified in the order of their vertices. If the figure be referred to AB, AC as rectangular axes of x, y, prove that the middle point of DE has coordinates $\frac{1}{2}(2b+c)$, $\frac{1}{2}(b+2c)$. Also find the equations of BG and CH, and prove that the join of their intersection to A is perpendicular to BC.

37. If P, Q, R are the middle points of the sides DE, FG, HK of the squares described, as in Example 36, on the sides of the right-angled triangle ABC, prove that

$$4\triangle PQR=2BC^2+13\triangle ABC.$$

38. A and B are two fixed points and C is a variable point. $ACDE$ and $BCFG$ are squares described on AC, BC external to the triangle ABC. Prove that M, the middle point of EG, is a fixed point, so long as C keeps to the same side of the line AB.

If the figure is referred to rectangular axes so that A and B are the points (p, q) and (r, s), what are the coordinates of M?

39. Show that there is a point such that the perpendicular from it on the line

$$x\sin^2\phi+2y\cos\phi=a$$

is the same whatever the value of ϕ, and find the coordinates of the point.

40. Find the ratio in which the line joining (1, 2) and (3, 1) is cut by the line $2x-3y+1=0$ and the coordinates of the point of intersection.

41. If AB, CD cut at O and A, B, C, D have coordinates $(-4, 1)$, $(2, 5)$, $(6, -3)$, $(-1, 4)$ respectively, find $AO : OB$, $CO : OD$, and the coordinates of O.

42. Prove that the line joining $(10, 5)$ and $(2, 3)$ is divided internally and externally in the same ratio by the circle $x^2+y^2=35$.

43. Find the equation of the parallel to $2x-3y+1=0$ through the image of $(1, -1)$ in the given line.

44. A, B, C, D are the four points $(2, 4)$, $(5, 1)$, $(10, 6)$, $(7, 3)$ respectively. Show how to draw a line through the origin dividing AB and CD in the same ratio. Prove that there are two solutions of the problem, and find the equation of each of the two lines which solve it.

45. Given one side of a quadrilateral in magnitude and position, the length of the opposite side, the angle between these two sides if produced, and the area of the quadrilateral, find the locus of one of the free vertices.

46. Find the equations of the two straight lines through the point $(3, 4)$, which are equidistant from the two points $(2, 1)$ and $(1, 2)$.

47. A and B are points on the x-axis, C and D are points on the y-axis, AEG and BFH are parallels to the y-axis, CEF and DGH are parallels to the x-axis, and AH and DE meet in P. If F is the point (a, b) and G the point (c, d), find the coordinates of P in terms of a, b, c, d, and prove that P lies on the line OF when O is the origin.

48. Find the equation of the line joining the point $(-2, 1)$ to the intersection of $x-y-1=0$ and $7x+y+33=0$, and prove that it bisects the angle between them.

49. A line moves so that the sum of its distances from the points $(5, -1)$, $(-3, -3)$ is equal to 12; find its envelope if the line does not pass between the points.

50. A line moves so that the sum of the reciprocals of the intercepts made on the axes is constant and equal to k. Prove that the line in all positions passes through the fixed point $(1/k, 1/k)$.

51. Prove that
$$x=\frac{a+bt}{p+qt}, \quad y=\frac{c+dt}{p+qt},$$
where t is a parameter, are freedom equations of a straight line, and find a transformation that would bring them into the form
$$x=a'+b'u, \quad y=c'+d'u,$$
where u is the parameter.

52. Find the in-centre of the triangle whose sides are
$$3x+4y-12=0, \quad 3x-4y=0, \quad y-6=0.$$

53. One side of a square of side a drawn from the origin has a gradient $\tan\theta$; prove that the equations of the diagonals are

$$y(\cos\theta-\sin\theta)=x(\sin\theta+\cos\theta),$$
$$y(\sin\theta+\cos\theta)+x(\cos\theta-\sin\theta)=a.$$

54. Find the in-centre of the triangle whose sides are

$$x-y+1=0,\quad x+y-7=0,\quad x-3y+5=0.$$

55. Prove that a common tangent to two circles cuts the join of centres internally or externally in the ratio of the radii.

Find all the common tangents to the circles

$$x^2+y^2-3x-4y=0 \quad\text{and}\quad x^2+y^2-21x+90=0.$$

56. Find the equation of a straight line through (a, b) and making an angle α with the line $y=mx+c$.

Find the equations of the sides of the rectangle which has $(1, 2)$, $(3, 4)$ as coordinates of the extremities of one diagonal and whose other diagonal is parallel to $2x-3y=0$.

57. Find the coordinates of a point such that the line joining it to the point (f, g) is bisected at right angles by the line $lx+my+n=0$; and find the locus of the first point when the only restriction on the given line is that it shall pass through a fixed point.

58. A and B are the fixed points $(a, 1/a)$, $(b, 1/b)$, P a variable point $(t, 1/t)$; PA and PB meet the axes of x and y in M, M' and N, N' respectively. Prove that MM' and NN' are of constant lengths.

59. If A and P be two points on the axis Ox, B and Q two points on the axis Oy, A and B being fixed and P, Q varying in such a manner that

$$\frac{1}{OA}-\frac{1}{OP}=\frac{1}{OB}-\frac{1}{OQ},$$

show that PQ passes through a fixed point.

60. The vertices of a triangle lie on the lines

$$y=x\tan\theta_1,\quad y=x\tan\theta_2,\quad y=x\tan\theta_3,$$

the circumcentre being at the origin; prove that the locus of the orthocentre is the line

$$x(\sin\theta_1+\sin\theta_2+\sin\theta_3)-y(\cos\theta_1+\cos\theta_2+\cos\theta_3)=0.$$

61. The equations of the sides of a triangle are

$$3x+4y=12,\quad 5x-12y=20,\quad 24y-7x=72.$$

Find (1) the area of the triangle, (2) the coordinates of the in-centre.

62. Find the equations of the tangents from the point $(11, 3)$ to the circle $x^2+y^2=65$ and of those from the point $(4, 5)$ to the circle

$$2x^2+2y^2-8x+12y+21=0.$$

63. Find the equation of the circle inscribed in the triangle whose sides are

$$x=0,\quad y=0,\quad x/4+y/3=1.$$

64. Draw the loci whose equations are

$$x-y=1, \quad x^2-a^2=0, \quad (x-a)^2+(y-b)^2=0,$$
$$(x-1)(x-2)+(y-3)(y-4)=0.$$

65. If the coordinates of A be $(3, 0)$, those of B $(0, 3)$ and those of C $(-3, 0)$; and if D divide AB so that $AD=\frac{2}{3}AB$, and E divide BC so that $BE=\frac{2}{3}BC$, find the points in which DE cuts the coordinate axes.

66. ABC is a triangle; if the coordinates of A, B be $(9, 0)$, $(0, 9)$, and the lengths of AC, BC be 13, 5, find the coordinates of C.

67. The coordinates of P and Q are $(0, 5)$ and $(15, -4)$. If the point R, whose coordinates are $(5, a)$, lies on PQ, find the value of a and find in what ratio PQ is divided at R.

68. A and B are two points on the x-axis equidistant from the origin O, and ABC is an equilateral triangle. Show that a point which moves so that the sum of the squares of its distances from the sides of the triangle is $\frac{3}{2}OA^2$ describes a circle. Find the radius and the coordinates of the centre, and draw the circle.

69. The straight line $x=a+bt$, $y=c+dt$ meets the axes in P and Q; find the area of triangle OPQ, where O is the origin.

70. If $x/a+y/b=1$ intersects $x/4a=y/b$ and $x/a=y/4b$ in P and Q, find an expression for the length of PQ.

71. Prove that the circles

$$x^2+y^2+\lambda x=a^2 \quad \text{and} \quad x^2+y^2+\mu y=b^2$$

touch if
$$\lambda^2\mu^2+4(b^2\lambda^2+a^2\mu^2)-4(a^2-b^2)^2=0.$$

72. A and B are the points $(a, 0)$ and $(0, b)$; $OAPB$ is a rectangle and Q is the projection of P on AB. If A and B move so that

$$a^3+b^3=c(a^2+b^2),$$

where c is constant, show that the locus of Q is a straight line.

73. P, Q, R start simultaneously from the points A, B, C, whose coordinates are (a, a'), (b, b'), (c, c'). If their component velocities parallel to the x and y axes are l and l', m and m', n and n' respectively, find when P, Q, R are collinear.

74. Prove that the centres of the three circles

$$x^2+y^2-4x-2y-7=0, \quad x^2+y^2=2x, \quad 2x^2+2y^2+4y=3$$

are collinear.

75. A, B, C are the points $(1, 0)$, $(0, 1)$, $(1, 1)$ respectively, and O is the origin. A point P moves so that the product of its perpendicular distances to OA, BC is equal to the product of its perpendicular distances to OB, AC. Find the equation of the locus of P, and discuss the equation.

76. Sketch roughly the path of a point which moves so that its distance from $(1, 0)$ exceeds by unity its distance from the y-axis, and find the equation of the path.

77. Find the equation of the parabola whose focus is $(0, 3)$ and whose directrix is $y=1$.

78. Draw roughly the form of the parabola whose focus is $(2, 8)$ and whose directrix is $x=y$, and find the equation of the parabola.

79. Draw the form of the ellipse whose foci are $(-3, 0)$ and $(3, 0)$ when the sum of the focal distances of any point on it is 8. Find the equation of the ellipse.

80. A variable point moves so that the difference of its distances from the points $(0, 0)$ and $(3, 4)$ is 3; draw the form of the locus and find its equation.

81. The focus of a parabola is $(3, 5)$ and the directrix is $x-2y=2$; find the equation of the parabola.

82. An ellipse has eccentricity $2/3$, a focus is $(-1, -4)$ and the corresponding directrix is $2x+3y=5$. Find the equation of the ellipse.

83. A hyperbola has a focus at the point $(2, 1)$, the corresponding directrix is $y=3x+5$ and the eccentricity is 2. Find the equation of the hyperbola.

84. A variable rectangle whose diagonal is of constant length a has one vertex at the origin O, a second vertex A on the x-axis and a third vertex B on the y-axis. If Q is the free vertex and $P(x, y)$ the projection of Q on AB, (1) find the equation of the locus of Q; (2) prove that

$$x/a=(OA/a)^3 \quad \text{and} \quad y/a=(OB/a)^3;$$

(3) find the equation of the locus of P, and sketch the form of the locus.

85. Find the equation of the radical axis and the length of the common chord of the circles

$$x^2+y^2+ax+by+c=0 \quad \text{and} \quad x^2+y^2+bx+ay+c=0.$$

86. Circles are drawn through the point $(c, 0)$ touching the circle $x^2+y^2=a^2$. Show that the locus of the pole of the axis of x with respect to these circles is the curve

$$4a^2(x-c)^4=(a^2-c^2)\{a^2-(c-2x)^2\}y^2.$$

87. Find the equation of the chord of contact of tangents to the circle $x^2+y^2=r^2$ from the point (h, k).

If this chord subtends a right angle at the point (h', k'), prove that

$$\frac{h'^2+k'^2-r^2}{hh'+kk'-r^2}=\frac{2r^2}{h^2+k^2}.$$

88. Find the coordinates of the limiting points of the circles

$$x^2+y^2-2x+8y+11=0 \quad \text{and} \quad x^2+y^2+4x+2y+5=0.$$

89. Find the equation of the circumcircle of the triangle formed by the pair of lines $x^2+2hxy-y^2=0$ and the line $y=mx+c$.

90. Find the equation of the pair of lines drawn (1) from the origin, (2) from the point (p, q) to the intersection of the line $lx+my+n=0$ with the line-pair

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

91. Prove that the circle which passes through the points $(at_1, a/t_1)$, $(at_2, a/t_2)$, $(at_3, a/t_3)$ also passes through the point $(a/t_1t_2t_3, at_1t_2t_3)$.

92. Find the equation of the circle circumscribing the square, two of whose adjacent sides are the lines joining the origin to the points $(a, 0)$, $(0, a)$. What is the equation of the tangent at the origin?

93. Prove that the two circles, each of which passes through the two points $(0, a)$, $(0, -a)$ and touches the straight line $y=mx+c$, will cut orthogonally if $c^2=a^2(2+m^2)$.

94. Prove that the circle on the line joining the origin to the point $(c^3, 1/c^3)$ as diameter passes through the point $(1/c, c)$.

95. Show that for a certain value of k the equation

$$(2x-y+3)(x-y+2)+k(3x-y+2)(3x-4y+2)=0$$

will represent a circle. Find the value; find also the radius and the coordinates of the centre of the circle.

96. The equations

$$x^2+y^2+\lambda(x-a)=0 \quad \text{and} \quad x^2+y^2+\mu(y-b)=0,$$

where λ, μ are parameters, represent two variable circles which touch one another; show that the locus of the point of contact is a circle, and find its equation.

97. The straight line joining the point (x_1, y_1) to the point (x_2, y_2) passes through the point $(c, 0)$ and subtends a right angle at the origin. If one point moves on the circle

$$x^2+y^2+2gx=0,$$

the other moves on the circle

$$c(x^2+y^2)+2g(x^2+y^2-cx)=0.$$

98. Find the condition that the line $x\cos\alpha+y\sin\alpha-p=0$ should touch the circle $x^2+y^2+2gx+2fy+c=0$.

Deduce the equation of the locus of the foot of the perpendicular from the origin on a variable tangent to the circle (called the **pedal** of the circle with respect to the origin).

99. The polars of a point P with respect to two given circles meet in Q; show that the radical axis of the circles bisects PQ.

100. Find the locus of a point such that the pair of lines joining it to the points $(-a, 0)$, $(a, 0)$ are harmonically conjugate with respect to the pair of lines joining it to the points $(0, -b)$, $(0, b)$.

MISCELLANEOUS EXAMPLES II.

1. Prove that the origin lies outside the circle

$$x^2+y^2-3x+2y+1=0.$$

Find the equations to the two tangents from the origin to the circle.

2. Two circles can be described to pass through the points (1, 2), (3, 4) and touch the line $x+2y-1=0$; find their equations.

3. The three sides of a variable triangle pass through three fixed points; one vertex lies on the line $x=0$, a second on the line $y=0$. Find the equation to the locus of the third.

4. The angular points of a quadrilateral taken in order are A, B, C, D. Points M, N are taken in AB and CD respectively, such that $AM:MB=CN:ND$; prove that the sum of the areas NAB and MCD is constant.

5. The straight lines joining a variable point P to two fixed points (x_1, y_1) and (x_2, y_2) meet the axis of x in M and N respectively. Find the equation to the locus of P if the ratio $OM:ON$ is given, O being the origin. Show in what cases the locus breaks up into straight lines, and give the geometrical explanation in each case.

6. Find the equations of the symmedians of the triangle whose vertices are the points (1, 0), (0, 2), (2, 4), and the coordinates of their point of intersection.

7. ACE and BDF are two straight lines. Show that the intersections of AB and DE, of BC and EF, of CD and FA, lie on a straight line, and find its equation referred to ACE, BDF as axes.

8. O is the origin, A a point whose coordinates are (2, 1), B a point whose coordinates are (3, 2); find the coordinates of a point P, chosen so that the triangles OPA, APB may be directly similar, the coordinate axes being supposed rectangular.

9. Show that the expression

$$\left(\frac{x}{a_1}+\frac{y}{b_2}-1\right)\left(\frac{x}{a_2}+\frac{y}{b_3}-1\right)\left(\frac{x}{a_3}+\frac{y}{b_1}-1\right)$$
$$-\left(\frac{x}{a_1}+\frac{y}{b_3}-1\right)\left(\frac{x}{a_2}+\frac{y}{b_1}-1\right)\left(\frac{x}{a_3}+\frac{y}{b_2}-1\right)$$

contains xy as a factor; and hence prove that if A_1, A_2, A_3 are three points on the axis of x, and B_1, B_2, B_3 three points on the axis of y, then the three points of intersection of A_1B_2 with A_2B_1, A_2B_3 with A_3B_2, and A_3B_1 with A_1B_3 lie on a straight line.

10. The equation to a certain straight line referred to rectangular axes is $Ax+By+C=0$; find its equation referred to the lines that trisect the angle between the axes.

11. Two circles of radii a and b touch the axis of y on opposite sides at the origin. The axes being rectangular, prove that the other two common tangents are given by

$$(b-a)x \pm 2\sqrt{ab} \,.\, y - 2ab = 0.$$

12. Prove that any two perpendicular straight lines passing through the points (x_1, y_1), (x_2, y_2) can be represented by the freedom equations

$$x = x_1 + t, \quad y = y_1 + \lambda t,$$

and

$$x = x_2 + u, \quad y = y_2 - u/\lambda,$$

respectively. Hence find the equation to the locus of the intersection of the above pair of lines, and interpret your result.

13. A and B are points on the axes of x and y respectively, such that $OA = a$, $OB = b$, $AB = c$; prove that the equation to the line joining the middle point of AB to the centre of the circle inscribed in the triangle OAB is

$$b(b+c-a)x - a(c+a-b)y + ab(a-b) = 0.$$

14. Deduce the equation of the bisectors of the angles formed by the line-pair $ax^2 + 2hxy + by^2 = 0$ by expressing the conditions that the line-pair $a'x^2 + 2h'xy + b'y^2 = 0$ should be (1) at right angles to each other, (2) harmonically conjugate with respect to the given line-pair.

15. If $(ABCD)$ and $(AB'C'D')$ are harmonic ranges, prove that BB', CC', DD' are concurrent.

16. Find the area of the triangle formed by the lines

$$lx + my = 1, \quad ax^2 + 2hxy + by^2 = 0.$$

17. Find the area of the triangle formed by the lines

$$y = m_1 x, \quad y = m_2 x, \quad ax + by + c = 0.$$

18. Find the equation of the straight line drawn in a given direction through the point of intersection of two given straight lines. Show that the coordinates of the orthocentre of the triangle formed by the lines

$$ax + by + c = 0, \quad bx + cy + a = 0, \quad cx + ay + b = 0$$

are given by

$$klmnx = abl + bcm + can, \quad klmny = cal + abm + bcn,$$

where $\quad 1/k = bc + ca + ab, \quad l = a^2 - bc, \quad m = b^2 - ca, \quad n = c^2 - ab.$

19. Draw the curve $\quad (x+y-1)^2 = 2(x-1)(y-1),$

and show how it is related to the lines

$$x + y - 1 = 0, \quad x - 1 = 0, \quad y - 1 = 0.$$

20. The equations of the sides of a triangle are

$$x + ly - l^2 = 0, \quad x + my - m^2 = 0, \quad x + ny - n^2 = 0;$$

find the coordinates of the orthocentre.

21. Prove that $$y^3-3x^2y=k(x^3-3xy^2),$$
where k is a parameter, represents any three equally inclined lines through the origin.

22. Prove that the coordinates of the in-centre of the triangle of sides a, b, c, whose opposite vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3), are
$$\frac{ax_1+bx_2+cx_3}{a+b+c}, \quad \frac{ay_1+by_2+cy_3}{a+b+c}.$$

23. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two straight lines, the equation may be written in any of the following forms:
$$(1)\ (h^2-ab)(ax+hy+g)^2-\{(h^2-ab)y+gh-af\}^2=0,$$
$$(2)\ (h^2-ab)(hx+by+f)^2-\{(h^2-ab)x+hf-bg\}^2=0,$$
$$(3)\ (g^2-ac)(gx+fy+c)^2-\{(g^2-ac)x+(fg-ch)y\}^2=0.$$

24. If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two straight lines, the lines are harmonically conjugate with respect to each of the following pairs of lines:
$$(1)\ ax+hy+g=0, \quad (h^2-ab)y+gh-af=0;$$
$$(2)\ hx+by+f=0, \quad (h^2-ab)x+hf-bg=0;$$
$$(3)\ gx+fy+c=0, \quad (g^2-ac)x+(fg-ch)y=0.$$

25. If (x_1, y_1) is the point of intersection of the line-pair
$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$
prove that
$$x_1=\frac{hc-gf}{af-gh}=\frac{f^2-bc}{bg-fh}=\frac{bg-fh}{h^2-ab},$$
$$y_1=\frac{hc-gf}{bg-fh}=\frac{g^2-ca}{af-gh}=\frac{af-gh}{h^2-ab}.$$

26. Prove that the equation
$$ax^2+2hxy+by^2+2gx+2fy+c=0$$
represents a pair of parallel straight lines if $h^2=ab$ and $af=gh$; and conversely, provided $a \neq 0$.

27. Prove that
$$(ax+by-1)(\alpha x+\beta y-1)+kxy=0$$
represents two straight lines if $k=(a-\alpha)(b-\beta)$; and find the coordinates of their point of intersection.

28. If $$ax^2+2hxy+by^2+2gx+2fy+c=0$$
and $$a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0$$
both represent line-pairs, prove that they will represent the same line-pair in two different positions, provided
$$(h^2-ab)(a'^2+b'^2)-(h'^2-a'b')(a^2+b^2)+2h^2a'b'-2h'^2ab=0.$$

29. O being the origin of rectangular axes, prove that if $px+qy+r=0$ cut $ax^2+2hxy+by^2=0$ in P and Q, then

$$OP.OQ=\frac{r^2\sqrt{\{(a-b)^2+4h^2\}}}{bp^2-2hpq+aq^2}.$$

30. A straight line AB of constant length has its extremities on two fixed straight lines OX, OY. Show that the locus of the orthocentre of triangle OAB is a circle.

31. Find the area of the quadrilateral bounded by the pairs of straight lines given by the equations

$$4x^2-3xy-y^2-2x+7y-6=0,$$
$$x^2+3xy-4y^2+7x-27y-44=0.$$

32. Show that the equation of the circle circumscribing the triangle formed by the lines

$$ax^2+2hxy+by^2=0 \quad \text{and} \quad px+qy-1=0$$

is $(x^2+y^2)(aq^2-2hpq+bp^2)+\{2hq+p(a-b)\}x+\{2hp-q(a-b)\}y=0.$

33. Investigate the equation to the polar of (x', y') with respect to the circle

$$x^2+y^2+2gx+c=0,$$

and show that, if g be a variable parameter and (x', y') a fixed point, then the polars of (x', y') with respect to the circles will pass through a fixed point lying on a circle through (x', y') and the limiting points of the circles.

34. Prove that

$$(a+2h+b)x^2+2(a-b)xy+(a-2h+b)y^2=0$$

denotes a pair of straight lines each inclined at an angle of 45° to one or other of the lines given by

$$ax^2+2hxy+by^2=0.$$

35. Find the equations of the three radical axes of the circles

$$(x-a)^2+(y-b)^2=b^2, \quad (x-b)^2+(y-a)^2=a^2,$$
$$(x-a-b-c)^2+y^2=ab+c^2,$$

and prove that they are concurrent. Find also the equation of the circle which cuts the three circles orthogonally.

36. Show that the equation

$$(ab-h^2)(ax^2+2hxy+by^2+2gx+2fy)+af^2+bg^2-2fgh=0$$

represents a pair of straight lines; and that these straight lines form a rhombus with the lines $ax^2+2hxy+by^2=0$, provided that

$$(a-b)fg+h(f^2-g^2)=0.$$

37. Find the equation of the circle which has for its diameter the chord cut off on the straight line $ax+by+c=0$ by the circle

$$(a^2+b^2)(x^2+y^2)=2c^2.$$

38. If the sides of a parallelogram be parallel to the lines

$$ax^2+2hxy+by^2=0,$$

and one diagonal be parallel to

$$lx+my+n=0,$$

show that the other diagonal is parallel to the line

$$y(bl-hm)=x(am-hl).$$

39. Show that the two lines given by

$$a(x^2+y^2)=(lx+my)^2$$

contain an angle $2\sin^{-1}\sqrt{\dfrac{a}{l^2+m^2}}$, and that $lx+my=0$ bisects one of the angles between the lines.

40. Show that the lines joining the origin to the intersections of

$$3x^2+5xy-3y^2+2x+3y=0 \quad \text{and} \quad 3x-2y=1$$

are at right angles.

41. The diagonals of a quadrilateral figure are represented by the equations $x=c$, $y=c$ and a pair of opposite sides by $ax^2+by^2=0$. Show that the other two sides intersect at the point $\{2cb/(b-a),\ 2ca/(a-b)\}$, and that they are parallel to the lines

$$(ax+by)^2+ab(x+y)^2=0.$$

42. Find the condition that the straight line $lx+my+n=0$ should touch the circle $(x-a)^2+(y-b)^2=r^2$.

Prove that the equations of the common tangents to the circle $x^2+y^2=289$ and the circle whose diameter is the chord of the first circle made by the line $x\cos\alpha+y\sin\alpha=15$ are

$$3(x\cos\alpha+y\sin\alpha)\pm 4(y\cos\alpha-x\sin\alpha)=85.$$

43. Prove that if the circle $x^2+y^2+2gx+2fy+c=0$ intercepts on the line $lx+my=1$ a length which subtends a right angle at the origin, then

$$c(l^2+m^2)+2(gl+fm+1)=0.$$

44. Find the equation to the line bisecting the acute angle between the lines joining the point (1, 1) to the points (1, 4) and (2, 3).

45. The points A, B, C are inverted into the points A', B', C' with respect to a circle of radius unity whose centre is the origin. If the coordinates of A, B, C are (6, 8), (3, 4) and (−3, 4), then

$$\triangle A'B'C'/\triangle ABC=1/1250.$$

46. Show that the straight line through (x_1, y_1) and the intersection of the lines $Ax+By+C=0$ and $A'x+B'y+C'=0$ is given by

$$\frac{Ax+By+C}{Ax_1+By_1+C}=\frac{A'x+B'y+C'}{A'x_1+B'y_1+C'},$$

and deduce the equation of the parallel through (x_1, y_1) to

$$Ax+By+C=0.$$

47. Find the gradient of the line $x=a+bt$, $y=c+dt$ by considering the gradient of the line joining the origin to the point at infinity on the given line.

48. The line $lx+my+n=0$ bisects an angle between a pair of lines of which one is $px+qy+r=0$; show that the other is

$$(px+qy+r)(l^2+m^2)-2(lp+mq)(lx+my+n)=0.$$

49. Show that the limiting points of the circle $x^2+y^2=a^2$ and an equal circle with centre on the line $lx+my=n$ lie on the curve

$$(lx+my)(x^2+y^2+a^2)=n(x^2+y^2).$$

50. If $$S\equiv x^2+y^2+2gx+2fy+c=0$$
and $$S'\equiv x^2+y^2+2g'x+2f'y+c'=0$$
are two circles of a coaxal system, show that the two point circles (or limiting points) of the system are given by the equation

$$S^2(f'^2+g'^2-c')-SS'(2ff'+2gg'-c-c')+S'^2(f^2+g^2-c)=0.$$

51. If OM, ON are the abscissa and ordinate of a point P with respect to rectangular axes $X'OX$, $Y'OY$, find the locus of P if $1/OM-1/ON=1$.

52. A locus is determined by the condition that the sum of the squares of the distances of any point P on it from two fixed points A and B is equal to the square of the distance of P from a straight line perpendicular to AB. Show that the equation to the locus can be put in the form $(x-a)^2+2y^2=b^2$.

53. Two points P and Q start from the same point A on the circumference of a circle and move along the circumference, Q moving twice as fast as P. Find a constraint equation for the locus of the intersection of the tangents at P and Q, and give a rough tracing of the locus.

54. OX, OY are rectangular axes and U, V fixed points whose coordinates are (p, q) and (r, s) respectively. A and B are points on the axes OX, OY respectively, and AU and BV meet in P. If the area of the quadrilateral $OAPB$ be constant, prove that the locus of P is a cubic curve which passes through the points (p, q), (r, s), (r, q).

55. A circle is described on OA as diameter, where O is the origin and A the point $(2a, 0)$. P is a variable point on this circle and OP is produced, either way, to Q so that $PQ=b$; find the equation of the locus of Q. The locus is called a limaçon.

56. A and B are fixed points on the axes of x and y, such that $OA=a$, $OB=b$. P is a movable point and BP and AP meet the axes of x and y in C and D. If the sum of the areas APC and BPD be constant and equal to c^2, find the equation to the locus of P.

57. A and B are fixed points on the axes of x and y respectively, such that $OA=a$, $OB=b$; A' and B' in like manner two other fixed points on the axes, such that $OA'=a'$, $OB'=b'$. A movable straight

line parallel to $A'B'$ meets the axes of x and y in A'' and B'' respectively. Prove that the locus of the intersection of $A''B$ and AB'' is a curve of the second degree which passes through O, A, B. What peculiarity arises when AB and $A'B'$ are parallel?

58. A and B are variable points on the axes of x and y respectively, the abscissa of A being p and the ordinate of B being q, such that $p+q=a$, a constant. A variable point P equidistant from A and B moves so that the area $OAPB=c^2$, a constant; find the equation of the locus of P.

59. A circle of radius a, whose centre is the origin, meets the axis of y in C. Q is a variable point on the circle, and OQ meets the tangent at C in T. M is the projection of Q on the axis of x, and MQ is produced to P so that $MP=CT$; find the equation of the locus of P, and roughly trace the locus.

60. OX, OY are fixed axes inclined at any angle; A and B are fixed points on OX and OY respectively, such that $OA=a$, $OB=b$. R is a point whose coordinates are (ξ, η), BR and AR meet OX and OY in P and Q respectively; find the equation to the straight line PQ, and show that if PQ moves so that $OP=OQ$, then the locus of R is a curve of the second degree which passes through O, A, B.

61. The vertex A of a triangle ABC is fixed, B moves on a fixed circle to which BC is a tangent and $CA=CB$; find the locus of C.

62. A is the point $(a, 0)$, Q is a variable point on the circle centre $(0, a/2)$, and radius $a/2$, whose abscissa and ordinate are ON, NQ. AQ meets the y-axis in R. If P is the point whose abscissa and ordinate are ON, OR, prove that the locus of P is

$$x^2y^2-2axy^2+a^2y^2+a^2xy+a^2x^2-a^3y=0.$$

Sketch the locus.

63. O is the origin, A is the point $(a, 0)$ and B the point $(b, 0)$, $(b<a)$; MQ is a variable ordinate of the circle on OA as diameter. If BP parallel to OQ meets MQ in P, sketch the locus of P, and prove that the equation of the locus is

$$xy^2+x^3-(2b+a)x^2+(b^2+2ab)x-ab^2=0.$$

64. Q is a variable point on the line $y=a$ which meets the y-axis at A. If O is the origin and on the line OQ are cut off QP, QP' equal to AQ, sketch the locus of P, P', and prove that the equation of the locus is

$$y^3-2ay^2+a^2y+x^2y-2ax^2=0.$$

If A is the origin, what does the equation become?

65. C is the point $(0, a)$; with centre C a circle is described to pass through the origin O. The ordinate through N, a variable point on the x-axis, meets the circle in Q. If P is a point on the ordinate NQ such that NP is a mean proportional between ON and NQ, sketch the locus of P, and prove that the equation of the locus is

$$x^4+y^4=2axy^2.$$

66. A is the point $(a, 0)$ and OC is a variable radius of the circle centre O, the origin, and radius a. B is a point on the circumference such that $AC=BC$, and from OC is cut off OP equal to the ordinate of B; sketch the locus of P, and prove that the equation of the locus is

$$(x^2+y^2)^3=4a^2x^2y^2.$$

67. O is the origin of rectangular axes and C is the point $(a, -a)$. The circle centre C, radius CO, is described, and ON, NQ are the abscissa and ordinate of a variable point Q on it. If P is a point on NQ such that $NP^2=ON \cdot NQ$, sketch the locus of P, and prove that the equation of the locus is

$$x^4+y^4=2ax(x^2-y^2).$$

68. O is the origin and C is the point $(-a, 0)$; the circle centre C, radius CO, is described. An equal circle rolls on this circle; sketch the locus of the point which is initially at O, and prove that the equation of the locus is

$$x^4+2x^2y^2+y^4+4ay^2x+4ax^3=4a^2y^2.$$

69. A variable ordinate meets the circles

$$(x-2a)^2+y^2=4a^2 \quad \text{and} \quad (x-4a)^2+y^2=16a^2$$

at Q and R. Sketch the locus of the middle point of QR, and prove that the equation of the locus is

$$y^4+x^2y^2-6axy^2+a^2x^2=0.$$

70. O is the origin; B, C are the points $(b, 0)$, $(c, 0)$ respectively. ON, NQ are the abscissa and ordinate of a variable point Q on the circle described on OB as diameter, and the parallel to OQ through C meets NQ in P; sketch the locus of P, and prove that the equation of the locus is

$$xy^2+x^3-(b+2c)x^2+(c^2+2bc)x-bc^2=0.$$

Examine the case when $c=b$.

CHAPTER X.

THE CONVERSE PROBLEM. GRAPHS OF EQUATIONS. POLYNOMIALS.

76. The Converse Problem. It has been shown how a specified locus or curve may be represented by an equation; the converse problem is to represent a given equation by its graph and to find the properties of the graph from the equation. Some cases have been already dealt with. Thus we can draw the graph of any equation of the forms

$$ax+by+c=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$ax^2+ay^2+2gx+2fy+c=0, \quad \ldots\ldots\ldots\ldots\ldots (2)$$

and we can find properties of the graphs by discussing the equations.

For example, the equation $x^2+y^2-6x=0$ represents a circle of radius 3, with its centre at the point (3, 0); the y-axis is a tangent to the circle, the origin being the point of contact; other tangents are at once seen to be $y=+3$, $y=-3$, $x=6$, and so on.

In Chapter IX. some other curves were considered, their equations being found and some properties stated. We now go on to discuss more fully the drawing of curves from their equations; at every step the student will have to remember that a point lies on the graph of a given equation if and only if the coordinates of the point satisfy the given equation.

77. Plotting by Points. The most straightforward way of obtaining the graph of a given equation is to calculate the coordinates of a number of points, plot the points and draw a curve through them. The chief rule to be observed is, that the points obtained must be close enough together to enable us to be sure that we have found the general trend of the curve; as we proceed we shall find means of

reducing the necessary number of calculations. The student has doubtless had some previous practice in plotting simple curves from their equations, and we shall here only refer to one or two important types.

Fig. 67 is the graph of $y=x^2$ from $x=-2$ to $x=2$.

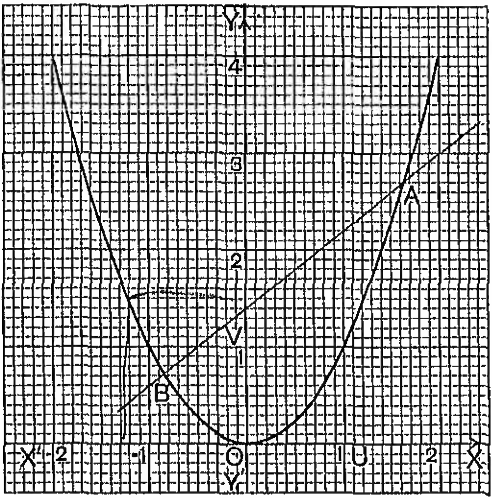


FIG. 67.

The diagram from which the figure is reproduced was drawn to a scale, "1 inch = 1," for both axes, and therefore shows only a small portion of the curve. As x increases beyond 2, the value of y grows very rapidly, and the curve rises rapidly; to show the curve for such values of y, we must choose a small unit for the y-axis.

In order to have clear notions of the way in which y varies as x varies, it is well to take the difference between successive values of x to be small, say 0·1, as shown in the following table:

x	0	±0·1	±0·2	...	±1	±1·1	±1·2	...	±2
y	0	0·01	0·04	...	1	1·21	1·44	...	4

The student should complete the table; it will be seen that when x increases by small amounts y also increases by small amounts, so that the curve is bound to be a continuous, unbroken line.

The curve is **symmetrical** about OY because, if a is any number, the y of the point whose x is $-a$ is the same as the y of the point whose x is $+a$; in other words, OY bisects all chords that are parallel to $X'OX$.

As a point moves along the curve from any position on the left of OY to any position on the right, the ordinate

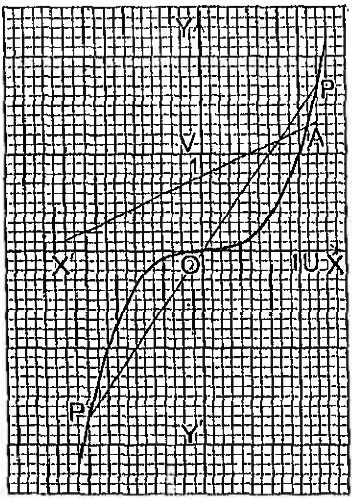


FIG. 68.

of the point decreases till the point reaches O, and then increases. The point O is therefore called a **turning point** of the graph; the value of the ordinate at the turning point—in this case, zero—is called a **turning value** of the ordinate.

The x-axis is a tangent to the curve at O.

The curve is a **parabola** (§ 70).

Fig. 68 is the graph of $y=x^3$ from $x=-1{\cdot}3$ to $x=1{\cdot}8$; for larger values of x the values of y soon become large,

and the curve rises very steeply on the right and descends very steeply on the left. The curve has no turning point.

In this case the origin is a **centre of symmetry**; if any point P on the curve is joined to O and the joining line produced till it cuts the curve again at P', then $OP' = PO$. It is easy to see that if P is the point (a, a^3), then P' is the point $(-a, -a^3)$.

Again, the x-axis is a tangent to the curve at O, but to the right of O the curve lies above the tangent, while to the left of O it lies below. A point such as O, where a curve crosses its tangent and bends away from it in opposite directions on opposite sides of the point, is called a **Point of Inflexion**; the tangent at the point is called an **Inflexional Tangent**.

For positive values of x (along OP), the curve is said to be **concave upwards**; for negative values of x (along OP'), the curve is said to be **convex upwards**.

The graph of $y = x^4$ resembles that of $y = x^2$; near the origin it lies closer to the x-axis, it crosses that of $y = x^2$ at the points (1, 1) and (−1, 1), and then rises more rapidly.

The graph of $y = x^5$ resembles that of $y = x^3$; near the origin it lies closer to the x-axis, it crosses that of $y = x^3$ at the points (1, 1) and (−1, −1), and then rises more rapidly on the right and descends more rapidly on the left.

In order to appreciate the differences in the behaviour of these curves, the student should draw on the same diagram the graphs of $y = x^n$ for $n = 2, 3, 4, 5$ from $x = 0$ to $x = 1{\cdot}2$, taking 1 inch as unit length for both axes. He should also draw the graphs of the same equations from $x = 1$ to $x = 2$; in this case the unit length for the y-axis must be small, say 0·2 inch, the unit length for the x-axis being still 1 inch.

78. The Graph of $y = ax^n$. If a is positive, the graph of $y = ax^n$ is obtained from that of $y = x^n$ by multiplying each ordinate of the latter curve by a; if $a = 2$ we double each ordinate, if $a = \frac{1}{2}$ we halve each ordinate, and so on. The graph is thus of the same general character as that of $y = x^n$; it lies, if $x > 0$, above the latter when $a > 1$, below when $a < 1$.

If a is negative, say $a = -c$, where c is positive, the graph of $y = ax^n$, that is, of $y = -cx^n$, is the reflexion in the x-axis of the graph of $y = cx^n$; or, if we produce each ordinate of $y = cx^n$ its own length, downwards when the ordinate is positive, but upwards when the ordinate is negative, the ends of these ordinates will lie on the graph of $y = -cx^n$.

The important thing to note is that when a is positive the curves have the forms shown in Fig. 69 (a), (b), and that when a is negative, they have the forms shown in Fig. 69 (c), (d).

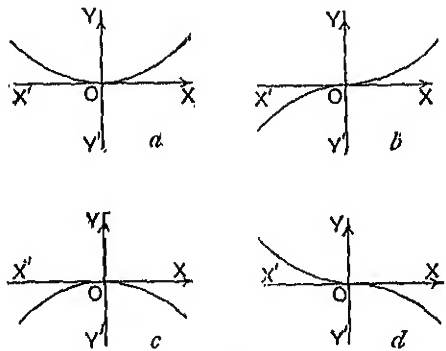


FIG. 69.

It is possible, by a mere change of scale, to interpret the graph of $y = x^n$ as being the graph of $y = ax^n$. For example, the graph of $y = x^2$ (Fig. 67) may be read as the graph of $y = 10x^2$ if the segment OV, which is unit segment for the graph of $y = x^2$, be taken as representing 10 units; in other words, let the unit segment of the y-axis be $\frac{1}{10}$th of the old unit segment, and the curve will become the graph of $y = 10x^2$. Similarly, Fig. 68 will be the graph of $y = \frac{1}{2}x^3$ if $OV = \frac{1}{2}$, that is, if the new unit segment of the y-axis be double the old unit segment.

These graphs are usually called **parabolic curves** of order n.

79. The Graph of $y^n = x$. If we interchange x and y, the equation $y = x^n$ becomes $y^n = x$. The graph of $y^n = x$ is therefore the same curve as the graph of $y = x^n$, but the axes do not occupy the usual positions; in place of

X, Y, X', Y', we must write Y, X, Y', X'. It will be an interesting exercise for the student to show that, when the scale units for the x-axis and the y-axis are the same, the graph of $y^n = x$ is the reflexion of the graph of $y = x^n$ in the bisector of the angles XOY, $X'OY'$; in this case we keep the old axes and draw a new curve.

When n is a positive integer the y-axis is a tangent at O to the graph of $y^n = x$.

The graph of $y^n = ax$ is of the same general character as that of $y^n = x$.

The student should work several of the following examples in order to become quite familiar with the curves; it will be sufficient in most cases to get the general shape correctly. He should, however, try to find the position of the turning points as accurately as possible. The transformations of Example 5 should be noted.

EXERCISES XX.

1. Trace, from $x = -2$ to $x = 2$, the graphs of

(i) $y = -x^2$; (ii) $y = -x^3$; (iii) $y = -x^4$; (iv) $y = -x^5$.

2. Trace the graph of $y = 2x^2$ from $x = -2$ to $x = 2$; take the scale unit of the x-axis to be 1 inch, that of the y-axis to be $\frac{1}{2}$ inch, and compare with Fig. 67.

3. The same as Example 2 for $y = 2x^3$, comparing the result with Fig. 68.

4. Trace, from $x = -2$ to $x = 2$, the graphs of

(i) $y = -3x^2$; (ii) $y = -3x^3$; (iii) $y = -3x^4$; (iv) $y = -3x^5$.

How could these graphs be obtained from those of Example 1?

5. If in Fig. 67 the line through $(0, -1)$, parallel to $X'OX$, be chosen as a new x-axis, what will be the equation of the graph? What will be the equation if the new x-axis is 1 unit above the old x-axis? What will be the equation if the new x-axis is the line through $(0, b)$ parallel to the old x-axis?

6. Sketch roughly for values of x between -2 and 2 the graphs of the following equations:

(i) $y = 2x^2 + 1$;	(ii) $y = 2x^2 - 1$;	(iii) $y = 2x^2 + 5$;
(iv) $y = 2x^2 - 5$;	(v) $y = -2x^2 + 1$;	(vi) $y = -2x^2 - 1$;
(vii) $y = -2x^2 + 5$;	(viii) $y = -2x^2 - 5$;	(ix) $y = 2x^3$;
(x) $y = 2x^3 + 3$;	(xi) $y = 2x^3 - 3$;	(xii) $y = -2x^3$;
(xiii) $y = -2x^3 + 3$;	(xiv) $y = -2x^3 - 3$.	

7. Graph the following equations:

(i) $y=\pm\sqrt{x}$; (ii) $y=\sqrt[3]{x}$; (iii) $y=-\sqrt[3]{x}$;

(iv) $y^2=-x$; (v) $y^2=x+4$; (vi) $y=\pm\sqrt{(9-x)}$.

80. Solution of Equations. Graphs are often useful for obtaining approximations to the roots of equations, and we shall take one or two examples.

Ex. 1. Find to 2 decimal places the roots of the equation

$$5x^2-4x-7=0. \quad \text{..............................(1)}$$

Write the equation in the form

$$x^2=0{\cdot}8x+1{\cdot}4, \quad \text{..............................(2)}$$

and consider the graphs of

$$y=x^2, \text{......(i)} \qquad y=0{\cdot}8x+1{\cdot}4. \text{......(ii)}$$

The graphs of (i) and (ii) are the curved line and the straight line of Fig. 67, which intersect at the points A and B. Now, the point A is on the graph of (i), and therefore, denoting by x_A, y_A the coordinates of A, we have

$$y_A=x_A^2.$$

But A lies also on the graph of (ii), and therefore

$$y_A=0{\cdot}8x_A+1{\cdot}4.$$

Hence $$x_A^2=0{\cdot}8x_A+1{\cdot}4;$$

in words, x_A is a root of equation (2), and therefore also of equation (1).

In the same way we see that x_B is a root of equation (1).

The values of x_A and x_B can be read off the figure, the accuracy of these values being determined partly by the care with which the graphs are drawn and partly by the scale of the graphs. We can read with fair accuracy to 2 decimals, and thus get

$$x_A=1{\cdot}65, \quad x_B=-0{\cdot}85.$$

These roots can of course be obtained more accurately by solving equation (1) algebraically; we are, however, concerned chiefly with a *method* which can be applied generally.

Ex. 2. Find to 2 decimals the roots of the equation

$$13x^3-6x-10=0. \quad \text{..............................(1)}$$

Write the equation in the form

$$x^3=\tfrac{6}{13}x+\tfrac{10}{13}=0{\cdot}46x+0{\cdot}77, \quad \text{........................(2)}$$

and consider the graphs of

$$y=x^3, \text{......(i)} \qquad y=0{\cdot}46x+0{\cdot}77, \text{......(ii)}$$

which are given in Fig. 68. These intersect only at the point A, and we have

$$y_A=x_A^3, \quad y_A=0{\cdot}46x_A+0{\cdot}77,$$

and therefore $$x_A^3=0{\cdot}46x_A+0{\cdot}77,$$

so that x_A is a root of equation (2), and therefore also of equation (1).

The value of x_A is 1·08 to 2 decimal places.

For equations of the form $ax^n+bx+c=0$ this method of solution is very convenient because the graph of x^n can be drawn with considerable accuracy by merely plotting a sufficient number of points. When the parts of the curve where the straight line intersects it have been approximately found by means of a rough sketch of curve and line, it is advisable to plot the curve carefully in the neighbourhood of the points of intersection; this can be done by finding two or three points in these neighbourhoods. So far as the solution of the equation is concerned, these neighbourhoods are the only parts of the curve wanted.

Ex. 3. Solve graphically to 2 decimal places the equation

$$x^2-2x-1=0. \qquad\qquad (1)$$

We take this example merely to illustrate another method. First draw the graph of the equation

$$y=x^2-2x-1. \qquad\qquad (2)$$

This equation is of such a simple kind that we can graph it fairly accurately by plotting points; we take the range of x from $x=-1$ to $x=3$ and draw up the following table:

x	-1	$-0{\cdot}8$	$-0{\cdot}6$	$-0{\cdot}4$	$-0{\cdot}2$	0	$0{\cdot}2$	$0{\cdot}4$	$0{\cdot}6$	$0{\cdot}8$	1
y	2	$1{\cdot}24$	$0{\cdot}56$	$-0{\cdot}04$	$-0{\cdot}56$	-1	$-1{\cdot}36$	$-1{\cdot}64$	$-1{\cdot}84$	$-1{\cdot}96$	-2

x	$1{\cdot}2$	$1{\cdot}4$	$1{\cdot}6$	$1{\cdot}8$	2	$2{\cdot}2$	$2{\cdot}4$	$2{\cdot}6$	$2{\cdot}8$	3
y	$-1{\cdot}96$	$-1{\cdot}84$	$-1{\cdot}64$	$-1{\cdot}36$	-1	$-0{\cdot}56$	$-0{\cdot}04$	$0{\cdot}56$	$1{\cdot}24$	2

We have calculated a large number of values because we are going to use the curve to solve several equations, and we must make certain that the curve is fairly accurate for the complete range chosen. If we wanted to find merely the general character of the curve, such a large number of points would not be required; even for the solution of equations we only need the parts of the curve near the points where it is met by other curves, and careful plotting near these points is in most cases quite sufficient.

Plot these points, taking each scale unit to be 1 inch, and draw a smooth curve through them (Fig. 70). The point $(1, -2)$ is the turning point.

Now, if P is any point on the graph, we have

$$y_p=x_p^2-2x_p-1. \qquad\qquad (3)$$

To solve equation (1) we have only to find those points on the graph of (2) for which the y is zero; A and B are the points, and therefore

$$0=x_A^2-2x_A-1, \quad 0=x_B^2-2x_B-1.$$

The roots of (1) are thus

$$x_A=2{\cdot}41, \quad x_B=-0{\cdot}41.$$

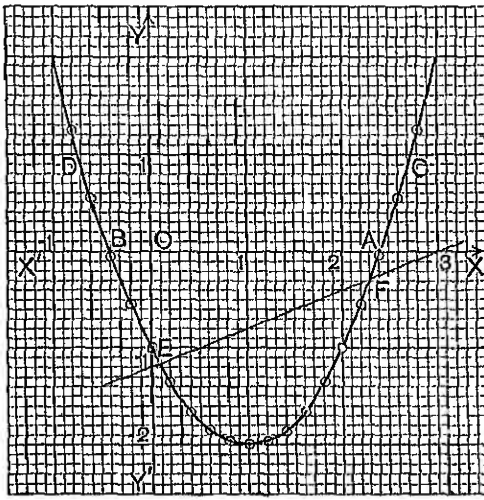


FIG. 70.

Again, to solve the equation $x^2-2x-2=0$, write it in the form

$$1=x^2-2x-1.$$

In equation (3) we must now have $y_P=1$. The two points on the curve for which the y is 1 are C and D, and therefore the required roots are

$$x_C=2{\cdot}73 \quad \text{and} \quad x_D=-0{\cdot}73.$$

Ex. 4. Use Fig. 70 to find to 2 decimals the roots of the following equations, and verify your results by algebraical solution:

(i) $x^2-2x-2{\cdot}5=0$; (ii) $x^2-2x-3=0$; (iii) $2x^2-4x-3=0$;
(iv) $2x^2-4x-5=0$; (v) $3x^2-6x-8=0$; (vi) $5x^2-10x-14=0$.

Ex. 5. Solve graphically the simultaneous equations

$$y=x^2-2x-1 \ \ldots\ldots\ldots\ldots(\text{i}), \quad 2x-5y=6 \ \ldots\ldots\ldots\ldots(\text{ii}).$$

On the diagram (Fig. 70) that contains the graph of equation (i), draw, with the same scale units, the straight line EF which is the

graph of equation (ii). Now if P is any point on the graph of (i) and Q any point on the graph of (ii), we have

$$y_P = x_P^2 - 2x_P - 1, \quad 2x_Q - 5y_Q = 6.$$

These two equations will be simultaneous if the points P and Q coincide; but the straight line intersects the curved line at E and F, and therefore

$$y_E = x_E^2 - 2x_E - 1, \quad 2x_E - 5y_E = 6.$$

Therefore x_E, y_E is one pair of solutions of equations (i) and (ii); similarly x_F, y_F is the other pair. Reading off the values of these coordinates, we find that the solutions are

$$x = 0\cdot09,\ y = -1\cdot17 \quad \text{and} \quad x = 2\cdot31,\ y = -0\cdot27.$$

Ex. 6. Solve graphically the simultaneous equations

(i) $y = x^2 - 2x - 1$, $x - y = 1$; (ii) $y = x^2 - 2x - 1$, $2x + 6y = 5$;

(iii) $y = x^2 - 2x - 1$, $2x - y = 5$; (iv) $y = x^2 - 2x - 1$, $2x + y + 2 = 0$.

Ex. 7. Solve graphically the equations

(i) $x^3 - 5x - 1 = 0$; (ii) $x^3 + 5x - 4 = 0$; (iii) $x^4 - 7x - 5 = 0$;

(iv) $2x^3 - 7x + 3 = 0$; (v) $8x^3 + 15x - 30 = 0$; (vi) $5x^4 - 27x - 10 = 0$.

Ex. 8. Solve graphically the simultaneous equations

(i) $x^2 + y^2 - 2x = 3$, $y = x^2$; (ii) $x^2 + y^2 - 4x - 2y = 20$, $y^2 = 4x$.

Ex. 9. Apply Example 8 to solve the equations

(i) $x^4 + x^2 - 2x = 3$; (ii) $(x^2 - 20)^2 = 16x$.

81. Function of x. The graph of the equation

$$y = x^2 - 2x - 1$$

is often called the graph of the function $x^2 - 2x - 1$. In calling $(x^2 - 2x - 1)$ a function of x we simply mean that $(x^2 - 2x - 1)$ depends for its value on the value of x, and varies when x varies. This variation is exhibited to the eye in the graph of Fig. 70. For example, we can say at once that $(x^2 - 2x - 1)$ is negative so long as x lies between $-0\cdot41$ and $2\cdot41$, that it is positive so long as x is outside these limits, that it vanishes when $x = -0\cdot41$ and when $x = 2\cdot41$, that it reaches its least value, namely -2, when $x = 1$, that its value is $-0\cdot8$ when $x = 2\cdot1$, and so on.

If y then stands for $(x^2 - 2x - 1)$ or if $y = x^2 - 2x - 1$, y is called a function of x; x and y are called **variables.** Here the variable y depends for its value on the value of the variable x; x is therefore called the **independent** variable

and y the dependent variable. Sometimes the independent variable x is called the argument of the function y.

The student will readily recall examples of two variables which are connected as independent and dependent. Thus the space described by a falling body is a variable, the time of falling is a variable, and the space described varies with the time of falling; we say the space described is a function of the time of falling, and this dependence is expressed by the equation $s=\frac{1}{2}gt^2$. Again the volume of a sphere varies when its radius varies; the volume is a function of the radius. What function is the volume V of the radius r? It is the function $\frac{4}{3}\pi r^3$, and we write $V=\frac{4}{3}\pi r^3$.

In these examples the independent variables are t and r, the dependent s and V respectively. We might ask, how does the time of fall vary with the distance fallen? The answer would be expressed in the equation $t=\sqrt{\left(\frac{2s}{g}\right)}$. In this case t depends for its value on the value of s; t is now the dependent and s the independent variable. That variable whose values are the objects of inquiry or calculation is called the dependent variable, the other being the independent variable.

We have then the following definition:

Definition. If two variables denoted by x and y are such that y varies in value when x varies in value, and if, when a value is assigned to x, the corresponding value of y can be determined, we say that y is a function of x.

The notation $f(x)$ is used to indicate a function of x, so that when y is a function of x we write $y=f(x)$. The letter f is a functional symbol, not a multiplier, and the symbol $f(x)$ must be taken *as a whole*. Other letters than f may be used, as $g(x)$, $F(x)$, $\phi(x)$, ..., and when different functions occur in the same problem different letters must be used.

The symbol $f(a)$ means "the value of the function $f(x)$ when x has the value a" or "the value of the function $f(x)$ when a is put in place of x." Thus if $f(x)=x^2-2x-1$,

$$f(a)=a^2-2a-1;\ f(3)=2;\ f(0)=-1,\ f(-1)=2.$$

We have noted above that the equation $s=\frac{1}{2}gt^2$ not only determines s when t is given, but also determines t when s is given. In general, the equation $y=f(x)$ not only determines y when x is given, but determines x when y is given; if the equation were solved for x in terms of y, we should find

$$x=\text{expression containing } y=F(y), \text{ say.}$$

In other words, an equation containing x and y enables us either to express y in terms of x or to express x in terms of y; an equation in x and y is therefore said *to define two functions* that are said to be **inverse** to each other.

For example, the equation $y=x^3$ gives, when solved for x, the equation $x=\sqrt[3]{y}$; it therefore defines the two functions, *the cube* of a variable and **the cube root** of a variable.

The following examples illustrate one or two technical terms.

Ex. 1. The equation $3xy-4x-5y+7=0$ defines two functions; state the functions explicitly.

Solving the equation for y in terms of x, we find

$$y=\frac{4x-7}{3x-5}=f(x). \quad \text{..............................(i)}$$

We have now expressed y explicitly as a function of x. If we solve for x in terms of y, we get

$$x=\frac{5y-7}{3y-4}=F(y), \quad \text{...........................(ii)}$$

and we have now expressed x explicitly as a function of y. In (i) the independent variable is x, while in (ii) the independent variable is y; the two functions $f(x)$, $F(y)$ are inverse functions.

Ex. 2. The equation $x^2-2xy+1=0$ defines two functions; state the functions explicitly.

Solving the equation for y in terms of x, we find

$$y=\frac{x^2+1}{2x}. \quad \text{..................................(i)}$$

Solve the equation for x in terms of y, and we now get

$$x=y+\sqrt{(y^2-1)} \quad \text{or} \quad x=y-\sqrt{(y^2-1)}. \quad \text{...............(ii)}$$

In this case, to each value of y (when y^2 is not less than 1) correspond *two* values of x; the function x defined by the given equation is

therefore said to be a *two-valued* function of y. The two sets of values of x belong, in the graphical representation, to two different parts of the one curve. For example, if we take the equation $y=x^2$ and solve for x, we find $x=+\sqrt{y}$ or $x=-\sqrt{y}$, and x is a two-valued function of y. The equation $x=+\sqrt{y}$ is represented by the curve OA to the right of the y-axis, and the equation $x=-\sqrt{y}$ by the curve OB to the left of the y-axis in Fig. 67, p. 183.

The given equation is said to define the functions *implicitly*; when it is solved and expressed as in (i) and (ii) the functions are defined *explicitly*.

Ex. 3. If y is the variable ordinate of a point Q which lies on the line joining the points A (0, 6) and B (3, 0), and if the line joining Q and the point C (0, 5) meets the x-axis in P so that the area of the triangle OPQ is a function $f(y)$ of y, prove that

$$f(y)=\frac{5y(y-6)}{4(y-5)}.$$

Since Q lies on the line AB, we have

$$2x_Q+y_Q=6. \quad\text{(i)}$$

The equation of CQ is

$$\frac{y-5}{x}=\frac{y_Q-5}{x_Q}. \quad\text{(ii)}$$

Now P lies on CQ and $y_P=0$; therefore, by (ii) and (i),

$$x_P=\frac{-5x_Q}{y_Q-5}=\frac{5(y_Q-6)}{2(y_Q-5)}. \quad\text{(iii)}$$

Hence, from (iii),

$$\triangle OPQ=\tfrac{1}{2}(x_Py_Q-x_Qy_P)=\frac{5y_Q(y_Q-6)}{4(y_Q-5)}.$$

Dropping the suffix, we have

$$f(y)=\frac{5y(y-6)}{4(y-5)}.$$

EXERCISES XXI.

1. AB is a straight line bisected at C. On AC, CB, AB are described semicircles all on the same side of AB. Let a circle, radius y, be described to touch the three semicircles. If $AC=x$, $y=f(x)$. Prove that $f(x)=x/3$.

2. A is the fixed point $(a, 0)$, referred to rectangular axes, origin O. The fixed circle, centre A, radius a, is described. A variable circle, radius y, is described in the quadrant XOY, to touch the fixed circle externally and to touch OX at P. If $OP=x$, $y=f(x)$. Prove that

$$y=\frac{x(x-2a)}{2a}.$$

3. ABC is a fixed triangle; P is a variable point in AB. PD, PE parallel to CA, CB respectively meet these sides in E, D respectively. If $AP=x$, $PD.PE=f(x)$, $PD/PE=F(x)$. Prove that

$$f(x)=\frac{abx(c-x)}{c^2} \text{ and } F(x)=\frac{b(c-x)}{ax};$$

where a, b, c are the sides of the triangle.

4. The diameter of a fixed circle is d. If $y=f(x)$, where x is the length of a variable chord and y the perpendicular distance of the chord from the centre, prove that $f(x)=\sqrt{(d^2-x^2)}$.

5. A is the point (0, 2) referred to rectangular axes, origin O, and the circle on OA as diameter is described, the centre being B. P is the variable point $(x, 0)$. The tangent from P to the circle meets the line $y=2$ in Q. If $PQ=f(x)$ prove that $f(x)=x+1/x$.

6. The readings on two thermometers, one Centigrade, the other Fahrenheit, immersed in a basin of water of uniform temperature, are y, x respectively. If $y=f(x)$ prove that $f(x)=\frac{5}{9}(x-32)$.

7. AB is the diameter of a fixed semicircle. P is a variable point in AB, and semicircles are described on AP, PB to lie within the fixed semicircle. A variable circle, radius y, touches the three semicircles; if $AP=x$, $y=f(x)$. Prove that $f(x)=\dfrac{ax(a-x)}{2(a^2-ax+x^2)}$, where $AB=a$.

8. A, B, C, D are the vertices in order of a quadrilateral, where $AB=CD=a$, a constant, $AC=BD=b$, a constant. If $BC=x$ and $AD=y$, then y is a function of x, say $f(x)$. Prove that $f(x)=(b^2-a^2)/x$.

82. Rough Form of a Graph. Polynomials. When we only wish to discuss the variation of a function in its leading features, it is desirable to be able to determine the shape of the graph rapidly. It is easy in certain cases to draw quickly the rough form of the graph; the following examples explain the method.

I. $$\mathbf{y=(x-1)(x-2).}$$

(1) Note the zeros of the function $(x-1)(x-2)$.

(a) When $x=1$, $y=0$. Mark A on the diagram (Fig. 71).

When x is a little less than 1, the factor $(x-1)$ is small, and therefore $(x-1)(x-2)$ is small and has the sign $(-)(-)$ or, on the whole, $(+)$. Mark B roughly on the diagram.

When x is a little greater than 1, the factor $(x-1)$ is small, and therefore $(x-1)(x-2)$ is small and has the sign $(+)(-)$ or $(-)$. Mark C roughly on the diagram.

(b) When $x=2$, $y=0$. Mark D on the diagram.

When x is a little less than 2, the factor $(x-2)$ is small, and therefore $(x-1)(x-2)$ is small and has the sign $(+)(-)$ or $(-)$. Mark E roughly on the diagram.

When x is a little greater than 2, the factor $(x-2)$ is small, and therefore $(x-1)(x-2)$ is small and has the sign $(+)(+)$ or $(+)$. Mark F roughly on the diagram.

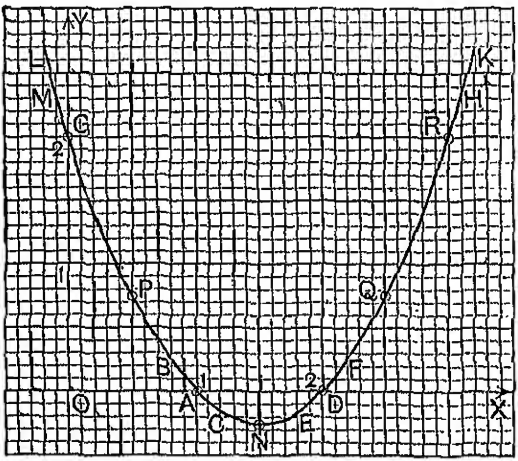


FIG. 71.

(2) Note the intercept on the y-axis.

When $x=0$, $y=2$. Mark G on the diagram.

(3) Examine the function when x is large.

When x is large and positive, $(x-1)(x-2)$ is large and positive. Mark HK roughly on the diagram.

When x is large and negative, $(x-1)(x-2)$ is large and positive. Mark LM roughly on the diagram.

(4) Mark selected points on the diagram to give what precision is desired to the trend of the graph.

When $x=1·5$, $y=-0·25$. Mark N.

When $x=0·5$, $y=0·75$. Mark P.

When $x=2·5$, $y=0·75$. Mark Q.

When $x=3$, $y=2$. Mark R.

(5) Finally join these parts of the graph by a smoothly running line. It is obvious that there must be a turning point between A and D, and it must be near $N(1·5, -0·25)$. At a later stage we shall be able to fix the position accurately.

All that need be written down preparatory to drawing the graph is the following table:

x	1	1−	1+	2	2−	2+	0	+∞	−∞	1·5	0·5	2·5	3
y	0	0+	0−	0	0−	0+	2	+∞	+∞	−0·25	0·75	0·75	2
giving	A	B	C	D	E	F	G	HK	LM	N	P	Q	R

The symbol 1− means a number a little less than 1, while 1+ means a number a little greater than 1; 2−, 2+, etc., have similar meanings.

Ex. Sketch the rough forms of the graphs of
(i) $y=(x-2)(x-3)$; (ii) $y=x(x-1)$; (iii) $y=(2-x)(x-1)$;
(iv) $y=-x(x-2)$; (v) $y=(x+2)(3-x)$; (vi) $y=(x+1)(x+2)$.

II. $\mathbf{y=x(x-1)(x-2)}$.

Construct the following table:

x	0	0−	0+	1	1−	1+	2	2−	2+	+∞	−∞	$\overline{0·5}$	0·5	1·5	2·5
y	0	0−	0+	0	0+	0−	0	0−	0+	+∞	−∞	$\overline{1·9}$	0·4	$\overline{0·4}$	1·9

giving A B C D E F G H K LM NP Q R S T of Fig. 72.

It is obvious that there must be a turning point between A and D, and another between D and G.

Ex. Sketch the rough forms of the graphs of

(i) $y=(x-1)(x-2)(x-3)$; (ii) $y=(x-2)(x-3)(x-4)$;
(iii) $y=(2-x)(x-1)(x-3)$; (iv) $y=-x(x-1)(x-2)$;
(v) $y=x(x+1)(x+3)$; (vi) $y=x(x^2-1)$.

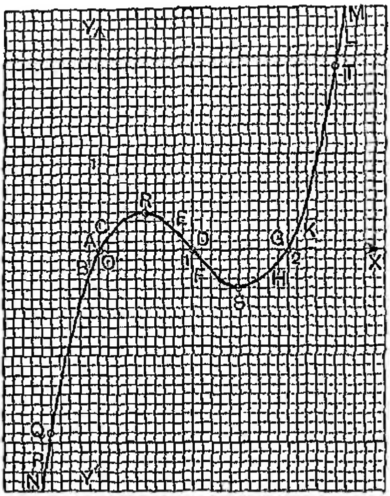


FIG. 72.

III. $y=x^2(x-2)$.

Construct the following table:

x	0	0−	0+	2	2−	2+	$+\infty$	$-\infty$	1	1·2	1·4	$\overline{0·5}$	2·2
y	0	0−	0−	0	0−	0+	$+\infty$	$-\infty$	$\bar{1}$	$\overline{1·2}$	$\overline{1·2}$	0·6	1

giving A B C D E F GH KL M N P Q R of Fig. 73.

Note the points M, N, P required to get even a rough notion of the dip of the graph. There is obviously a turning point between N and P.

83. Graph of $y^2=f(x)$. We consider some simple cases, $f(x)$ being a polynomial.

To each value of x there are two values of y which are numerically equal but are of opposite signs; the x-axis is therefore an axis of symmetry, so that in calculating co-ordinates we need only attend to one value of y.

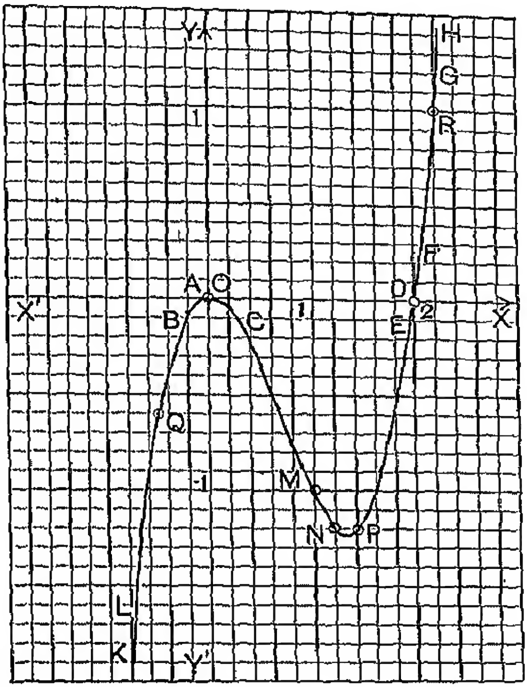


FIG. 73.

I. $y^2=x^3$ or $y=\pm\sqrt{(x^3)}$.

If x is negative y will be imaginary; there is no part of the curve, therefore, to the left of the y-axis. The values of $\sqrt{(x^3)}$ are easily calculated for a series of values 0, 0·1, 0·2, ... of x; the curve, which is called the **semi-cubical parabola**, is shown in Fig. 74. It consists of two

branches OA, OB, and the x-axis is a tangent at O to both branches.

A point such as O, at which two branches OA, OB have the same tangent, but *beyond which they do not pass*, is called a Cusp.

The graphs of $y^2=x^n$, where n is a positive *odd* integer greater than unity, have all a cusp at the origin, the x-axis

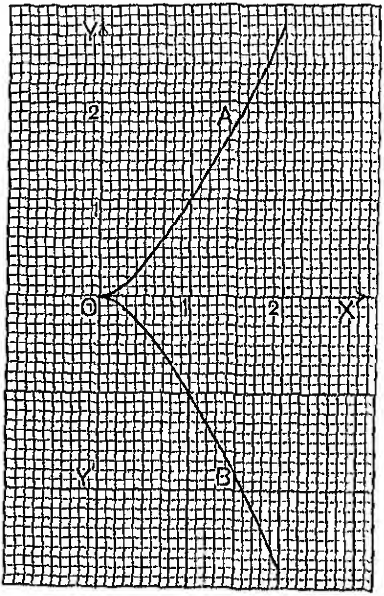


FIG. 74.

being the tangent there. The graph of $y^2=x$ has not a cusp.

II. $y^2=x(x-1)(x-2)$ or $y=\pm\sqrt{\{x(x-1)(x-2)\}}$.

First draw the graph of

$$y_1=x(x-1)(x-2);$$

then

$$y^2=y_1 \quad \text{or} \quad y=\pm\sqrt{y_1}.$$

The graph of y_1 is shown in Fig. 72.

If y_1 is negative, y will be imaginary; hence we need only consider the values of x that correspond to the part of the graph (Fig. 72) between $x=0$ and $x=1$ and the part from $x=2$ onwards. We see at once that the rough form of the graph consists of an oval lying between $x=0$ and

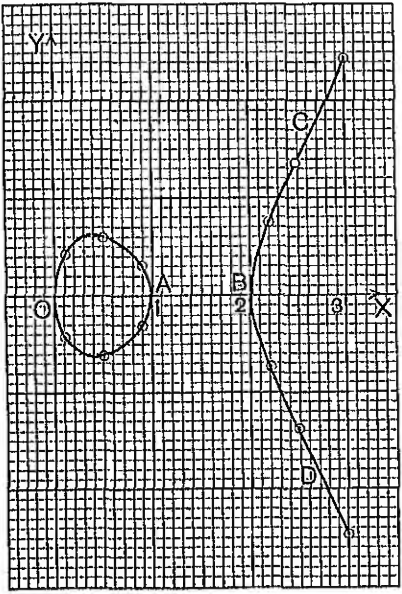


FIG. 75.

$x=1$ and an open branch from $x=2$ onwards. The curve (Fig. 75) is now drawn from the following table:

x	0	0·1	0·5	0·9	1	2	2·2	2·5	3
y_1	0	0·17	0·375	0·10	0	0	0·53	1·875	6
$\pm y$	0	0·41	0·61	0·32	0	0	0·73	1·37	2·45

III. $\quad y^2=x(x-1)^2 \quad$ or $\quad y=\pm(x-1)\sqrt{x}.$

If x is negative, y is imaginary. Let us take

$$y_1=(x-1)\sqrt{x},$$

and plot the part of the curve corresponding to this equation.

$y_1=0$ when $x=0$ and when $x=1$. When x is a proper fraction, y_1 is negative; when x is greater than unity, y_1 is positive, and y_1 now steadily increases as x increases. We give the table:

x	0	0·1	0·3	0·5	0·8	1	1·2	1·5	2	2·5	3
y_1	0	−0·28	−0·38	−0·35	−0·18	0	0·22	0·61	1·41	2·37	3·46

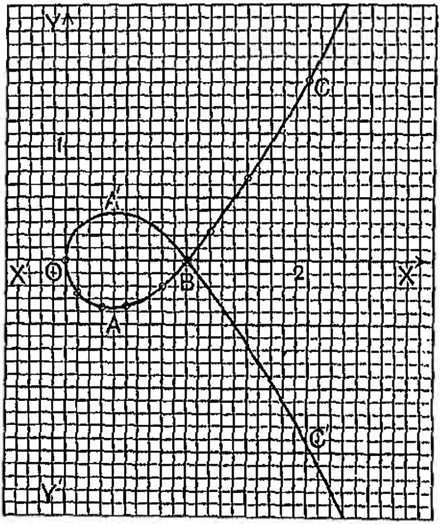


FIG. 76.

Plotting these points, we get the part $OABC$ of Fig. 76; the part $OA'BC'$ is the reflexion of $OABC$ in the x-axis, and may be obtained by plotting the points derived from the above table by changing the sign of each ordinate. The y-axis is a tangent.

It is obvious that as x increases from 0 to 1, y cannot possibly become as large as unity; in fact, the table

suggests that 0·4 will be an upper limit to the value of y. The curve, therefore, must have a loop such as $OABA'$.

A point such as B, at which two branches of a curve cross, is called a node.

The rough form of the graph may also be readily obtained by first graphing $y=x(x-1)^2$ by the methods of § 82 and then marking roughly the points whose ordinates are the square roots of the *positive* ordinates of this graph.

In the same way we see from Fig. 73 that the graph of $y^2=x^2(x-2)$ consists of the origin and an open branch from $x=2$ onwards. An isolated point, like the origin in this case, whose coordinates satisfy the equation of the curve but in whose neighbourhood there is no other point of the curve is called **a conjugate point** or an **isolated point.**

EXERCISES XXII.

Trace the rough forms of the graphs of the following equations, plotting a few chosen points to give some precision to the graphs:

1. $y=x^2+x-2$. 2. $y=(2x+1)(x-1)$. 3. $y=(x+1)(2-x)$.
4. $y=(2x+1)(1-x)$. 5. $y=(x+1)(x^2-1)$. 6. $y=x(x-1)^2$.
7. $y=x^2(x+1)$. 8. $y=x^2(2-x)$. 9. $y=-x(x+2)^2$.
10. $y=x+x^4$. 11. $y=x^2(x-1)(x-2)$. 12. $y=x^2(x^2-1)$.
13. $y=x(x+1)^2(x+2)$. 14. $y=x^3(x-1)$. 15. $y=(x-1)^2(x-2)^2$.
16. $y=x(x+1)^3$. 17. $y=(x-1)(x-2)(x-3)(x-4)$.
18. $y=x^2(x-1)^3$. 19. $y=x^3-x^6$. 20. $y=x^4(x-1)$.
21. $y^2=x(x-1)$. 22. $y^2=(x-1)(x-2)(x-3)$.
23. $y^2=(x-1)(x-2)^2$. 24. $y^2=x^2(1-x^2)$. 25. $y^2=x^3-x^5$.
26. Trace in the same diagram the graphs of
$$y^2=x^3 \quad \text{and} \quad y^2=x^5,$$
and derive from your graphs those of
$$y^3=x^2 \quad \text{and} \quad y^5=x^2.$$
27. Trace the graphs of the following equations:
(i) $y=\pm\sqrt{(1-x^2)}$; (ii) $y=\pm\sqrt{(x^2-1)}$;
(iii) $y=\pm\sqrt{(2ax+bx^2)}$ when $a=3$ and b has the values -4, -1 0, 4; (iv) $y=\pm\sqrt{\{(x-1)(x-2)\}}$; (v) $y=\pm\sqrt{\{(x-1)(x-3)\}}$;
(vi) $y=\pm\sqrt{\{(x-1)(5-2x)\}}$.

CHAPTER XI.

ROUGH GRAPHS OF RATIONAL FRACTIONS. FREEDOM EQUATIONS.

84. Graph of $y=\frac{2-x}{x-1}$. Construct the following table:

x	2	2−	2+	1	1−	1+	0	small	∞
y	0	0+	0−	∞	$-\infty$	$+\infty$	-2	$-2-x-x^2$ approx.	-1
	A	B	C	D, E	FG	HK	L	Note (i)	M, N

x	large	-2	1/2	1·4	3
y	$-1+1/x$ approx.	$-4/3$	-3	1·5	$-1/2$
	PQ, RS. Note (ii)	T	U	V	W

From this table Fig. 77 has been drawn.
The curve is a hyperbola.

Note (i). If an approximation to the value of a numerical fraction, say 237/1892, is wanted, we divide 237 by 1892, getting 0·1252..., and we retain one or more of the first figures 0·1, 0·12, 0·125, ..., according to the degree of accuracy required. In the given fraction the figures of numerator and denominator are arranged in the order of importance; thus 2 is the most important figure of the numerator and comes first, but in the denominator 2 is the least important and comes last. In the fraction $(2-x)/(x-1)$ the numerator is, when x is *small*, already arranged in the order of importance of its terms, but the denominator must be written in the form $-1+x$, so that the most important

term -1 may come first. If we divide $2-x$ by $-1+x$, the first few terms will give an approximation to the fraction when x is small. In general, when x is small the numerator and denominator of a fraction, arranged in *ascending* powers of x, will give by division an approximation to the fraction; the process is called **Ascending Continued Division.**

Carrying out the division of $2-x$ by $-1+x$ as indicated at the side, we find as an approximation

$$y=-2-x-x^2.$$

$$\begin{array}{r|l}
-1+x\ ) & 2-x\ (-2-x-x^2 \\
 & \underline{2-2x} \\
 & \quad x \\
 & \quad \underline{x-x^2} \\
 & \qquad x^2 \\
 & \qquad \underline{x^2-x^3} \\
 & \qquad\quad x^3
\end{array}$$

It will be seen that the exact value of y is

$$y=-2-x-x^2+\frac{x^3}{-1+x},$$

so that we can in any given case determine the degree of accuracy in the approximation.

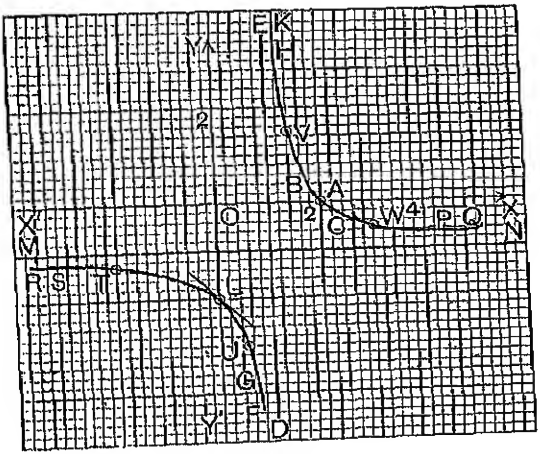


FIG. 77.

The straight line $y=-2-x$ is the *tangent* to the graph at L; near L the graph must be below the straight line on both sides of L, because $-2-x-x^2$ is less than $-2-x$ for

every small value of x, whether that value be positive or negative.

Note (ii). When x is large we write the fraction in the form $\dfrac{-x+2}{x-1}$, so that both in numerator and in denominator the most important term may come first, and, if we now divide, the first few terms of the quotient will give an approximation to the fraction when x is large. In this case the division must usually be continued till one term at least of the quotient contains x in a denominator. In general, when x is large the numerator and denominator of a fraction, arranged in *descending* powers of x, will give by division an approximation to the fraction; the process is called **Descending Continued Division.**

Carrying out the division as indicated at the side, we find that

$$\begin{array}{r|l} x-1 & -x+2\ \left(-1+\frac{1}{x}\right. \\ & \underline{-x+1} \\ & \quad +1 \\ & \quad \underline{+1-\frac{1}{x}} \\ & \qquad +\frac{1}{x} \end{array}$$

$$y=-1+\frac{1}{x}$$

is an approximation to the fraction when x is large.

When x is *positive*,

$$y=-1+\frac{1}{x}>-1;$$

therefore, on the far right (PQ in the diagram) the grap appears *above* the line MN, whose ordinate is -1.

When x is *negative*,

$$y=-1+\frac{1}{x}<-1;$$

therefore, on the far left (RS in the diagram) the grap appears *below* the line MN.

The approximation $y=-1+\dfrac{1}{x}$ shows that if a point trav along the curve past P, then Q and so on, the point w come closer and closer to the straight line MN; similarl if it travel past S, then R and so on. MN is called *asymptote* of the curve. DE is also an *asymptote*. T following is the formal definition of an asymptote.

Definition. A straight line MN is said to be an asymptote to a branch of a curve which goes off to infinity, when the distance of a variable point on the branch from the line MN tends towards zero, as the point moves to an infinite distance along the branch.

COR. 1. The graph of $x=\dfrac{2-y}{y-1}$ may be derived from the graph of $y=\dfrac{2-x}{x-1}$ by interchanging the axes.

COR. 2. The graph of $y=\dfrac{a(2a-x)}{x-a}$ (a positive) may be derived from the graph of $y=\dfrac{2-x}{x-1}$ by substituting the abscissae a, $2a$, etc., for the abscissae 1, 2, etc., and the ordinates a, $2a$, etc., for 1, 2, etc. Indeed, the graph of $y=\dfrac{2-x}{x-1}$ is the graph of $y=\dfrac{a(2a-x)}{x-a}$ reduced in the ratio $1:a$. If a were negative, what would be the relation of the graph of $y=\dfrac{a(2a-x)}{x-a}$ to that of $y=\dfrac{2-x}{x-1}$?

85. Graph of $y=\dfrac{x^3-3x^2+4}{x^2}$.

(1) Examine the zeros of the function. Put $y=0$;

then $$0=x^3-3x^2+4$$

or $$0=(x+1)(x-2)^2,$$

so that $y=0$ when $x=-1$ and when $x=+2$. Mark A, D on the diagram (Fig. 78).

Now $$y=\frac{(x+1)(x-2)^2}{x^2}. \quad\text{.....................(1)}$$

When $x=-1-$, $y=0-$. Mark B roughly on the diagram.
When $x=-1+$, $y=0+$. " C " "
When $x=\ \ 2-$, $y=0+$. " E " "
When $x=\ \ 2+$, $y=0+$. " F " "

(2) Examine the appearance of the graph when x is

small. If we start with equation (1), we must use Ascending Continued Division, and we should obtain the form

$$y=\frac{4}{x^2}-3+x \quad \text{or} \quad y=\frac{4}{x^2} \text{ approx.}$$

When $x=0$, $y=\infty$.

When $x=0-$, $y=+\infty$. Mark GH roughly on the diagram.

When $x=0+$, $y=+\infty$. " KL " "

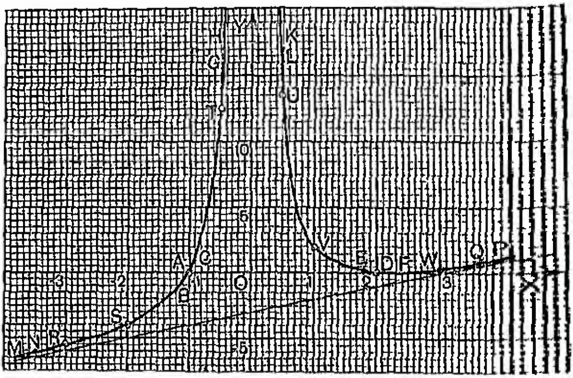


FIG. 78.

(3) Examine the appearance of the graph when x is large.

By Descending Continued Division applied to $\frac{x^3-3x^2+4}{x^2}$, we have

$$y=x-3+\frac{4}{x^2}.$$

Draw the line $y_1=x-3$.

When x is negative, $y=(x-3)+$. Mark MN roughly on the diagram.

When x is positive, $y=(x-3)+$. Mark PQ roughly on the diagram.

(4) The trend of the graph is now apparent. To give some precision to the graph, plot the points R, S, T, U, V, W. Draw a smooth line through the points A ... W.

All that needs to be written down is the following table:

x	-1	$-1-$	$-1+$	2	$2-$	$2+$	0	small	large	-3	-2	$-\frac{1}{2}$	$\frac{1}{2}$	1	3
y	0	$0-$	$0+$	0	$0+$	$0+$	∞	$4/x^3$	$x-3+\frac{4}{x^2}$	$-5{\cdot}6$	-4	$12\frac{1}{2}$	$13\frac{1}{2}$	2	$0{\cdot}4$
giving	A	B	C	D	E	F		GH, KL	MN, PQ	R	S	T	U	V	W

in Fig. 78.

COR. In place of the numbers 1, 2, etc., on the axes, write a, $2a$, etc., and the graph will be that of

$$y=\frac{x^3-3ax^2+4a^3}{x^2}.$$

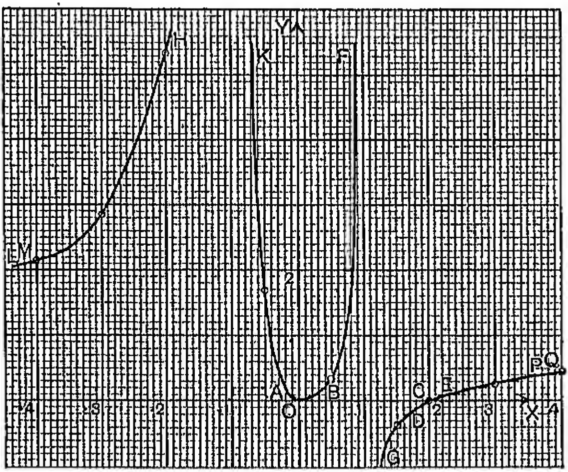


FIG. 79.

86. Graph of $y=\dfrac{x^2(x-2)}{(x+1)^2(x-1)}$.

The following table shows the general trend of the graph:

x	0	$0-$	$0+$	2	$2-$	$2+$	1	$1-$	$1+$	-1	$-1-$	$-1+$	large
y	0	$0+$	$0+$	0	$0-$	$0+$	∞	$+\infty$	$-\infty$	∞	$+\infty$	$+\infty$	$1-3/x$
giving	O	A	B	C	D	E		F	G		H	K	LM, PQ

of Fig. 79.

The plotted points

x	-4	-3	-2	-0.5	0.5	1.5	3	4
y	2.13	2.81	5.33	1.67	0.33	-0.36	0.28	0.43

give the graph as shown in Fig. 79.

87. Graph of $y^2=\dfrac{x^2(x-2)}{(x+1)^2(x-1)}$.

First draw the graph of

$$y_1=\frac{x^2(x-2)}{(x+1)^2(x-1)},$$

as in Fig. 79.

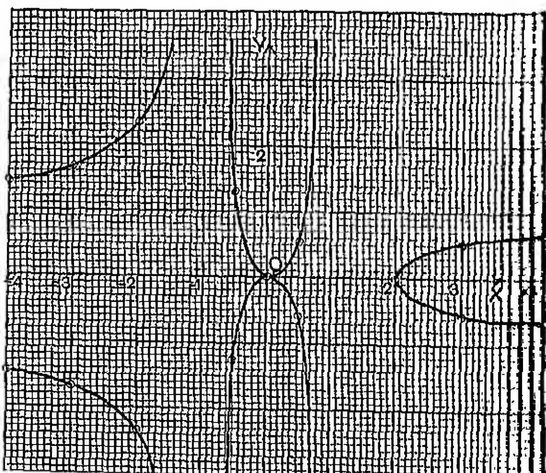


FIG. 80.

Then $y^2=y_1$ or $y=\pm\sqrt{y_1}$. For real values of y, y_1 must be positive, so that there is no part of the graph of y between $x=1$ and $x=2$.

The trend of the graph is easily seen. The plotted points

x	-4	-3	-2	$-1·5$	$-0·5$	0	$0·5$	2	3	4
y_1	2·13	2·81	5·33	12·00	1·67	0	0·33	0	0·28	0·43
y	$\pm 1·46$	$\pm 1·68$	$\pm 2·31$	$\pm 3·55$	$\pm 1·29$	± 0	$\pm 0·58$	± 0	$\pm 0·53$	$\pm 0·66$

then give Fig. 80.

88. Solution of Simultaneous Equations. The following example illustrates some methods of procedure.

Ex. Solve the simultaneous equations

$$x^2 - xy = 2, \quad \ldots\ldots\ldots\ldots(1) \qquad y^2 + xy = 3. \quad \ldots\ldots\ldots\ldots(2)$$

First, graph the equation $x^2 - xy = 2$ or $y = x - \dfrac{2}{x}$.

The table

x	$0+$	$0-$	$+$large	$-$large
y	$-\infty$	$+\infty$	$x-$	$x+$

gives the trend of the graph; the additional values

x	0·7	0·8	1	2	3	1·41
y	$-2·16$	$-1·7$	-1	1	2·33	0

give the graph as in Fig. 81. The asymptotes are given by the equations $y = x$ and $x = 0$.

Next, graph the equation $y^2 + xy = 3$ or $x = -y + \dfrac{3}{y}$.

The table

y	$0+$	$0-$	$+$large	$-$large
x	$+\infty$	$-\infty$	$-y+$	$-y-$.

gives the trend of the graph; the additional values

y	0·8	1	2	3	1·73
x	2·95	2	$-0·5$	-2	0

give the graph as in Fig. 81. The asymptotes are given by the equations $y+x=0$ and $y=0$.

The solutions, being the coordinates of the points of intersection of the graphs, are therefore

x	2	-2	0·71	$-0·71$
y	1	-1	$-2·13$	2·13

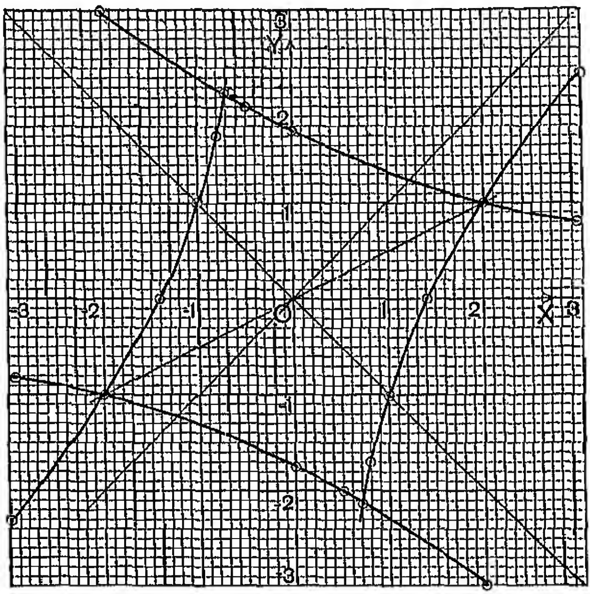


FIG. 81.

If we multiply equation (1) by 3 and equation (2) by 2, and then subtract, we get

$$3x^2-5xy-2y^2=0 \quad \text{or} \quad (3x+y)(x-2y)=0. \quad \ldots\ldots\ldots\ldots(3)$$

The intersections of $3x+y=0$, and either (1) or (2) give solutions. Also the intersections of $x-2y=0$ (shown in diagram), and either (1) or (2) give solutions. In fact, the solutions of (1) and (2) are the same as the solutions of

$$\begin{cases} x^2-xy=2 \\ 3x+y=0 \end{cases} \quad \text{and} \quad \begin{cases} x^2-xy=2 \\ x-2y=0 \end{cases}$$

$$\text{or of} \quad \begin{cases} y^2+xy=3 \\ 3x+y=0 \end{cases} \quad \text{and} \quad \begin{cases} y^2+xy=3 \\ x-2y=0 \end{cases}$$

EXERCISES XXIII.

Trace the rough graphs of the following equations, plotting a few points to give some precision to the graphs:

1. $y=\frac{1}{x}$. 2. $y=-\frac{1}{x}$. 3. $y=\frac{1}{x-1}$. 4. $y=\frac{x-1}{x-2}$.

5. $xy-x-y-1=0$, *i.e.* $y=\frac{x+1}{x-1}$. 6. $2xy-x-y+2=0$. 7. $x=\frac{y-1}{y-2}$.

8. $2xy-x+y+2=0$. 9. $y=1+\frac{1}{x}$.

10. $y=x+\frac{1}{x}$. 11. $y=2x-\frac{8}{x}$. 12. $y=x+3+\frac{2}{x}$.

13. $y=-2x-\frac{2}{x+2}$. 14. $y=x+\frac{1}{x^2}$. 15. $y=x-\frac{1}{x^2}$.

16. $y=x^2+\frac{1}{x}$. 17. $y=x^2-\frac{1}{x^2}$. 18. $y=x^3+\frac{1}{x^2}$.

19. $y=x^3-\frac{1}{x}$. 20. $y=x+x^2-\frac{2}{x}$.

21. $y=2x+3-\frac{5}{x^2}$. 22. $y=\frac{(x-1)(x-2)}{x-3}$.

[Prove that, in Ex. 22, $y=x+\frac{2}{x}$ is the approximation when x is large.]

23. $y=\frac{x-1}{(x-2)(x-3)}$. 24. $y=\frac{(x-1)^2}{x-2}$. 25. $y=\frac{x-1}{(x-2)^2}$.

26. $y=\frac{a}{x^2}$ (i) when a is positive, (ii) when a is negative.

27. $y=\frac{x^2(x-1)}{(x-2)^2}$. 28. $y=\frac{x^3+1}{x^3-1}$. 29. $y^2=\frac{x(x-2)}{x-1}$.

30. $y^2=\frac{x^2(x-1)}{(x+1)(x+2)}$. 31. $y^2=\frac{x(x-2)^2}{x^2-1}$. 32. $x=\frac{(y-1)(y-3)}{y-2}$.

33. $x=\frac{y^2(y-1)}{(y-2)^2}$. 34. $x^2=\frac{y^2(y-1)^2}{(y+1)(y^2-4)}$.

35. Graph on the same diagram the following equations:

$$\text{(i) } y^2=\frac{1}{x}, \quad \text{(ii) } y^2=\frac{1}{x^3}, \quad \text{(iii) } y^2=\frac{1}{x^5},$$

and use your graphs to derive those of

$$\text{(iv) } x^2=\frac{1}{y}, \quad \text{(v) } x^2=\frac{1}{y^3}, \quad \text{(vi) } x^2=\frac{1}{y^5}.$$

36. Graph the following equations:

$$\text{(i) } y^2=\frac{(x-1)(x-2)}{x-3}, \quad \text{(ii) } y^2=\frac{x-3}{(x-1)(x-2)}.$$

37. Trace the form of the Conchoid from its equation

$$y^2=\frac{x^2(b-c+x)(b+c-x)}{(x-c)^2}$$

(i) when $b<c$, (ii) when $b=c$, (iii) when $b>c$.

38. Trace the form of the Cissoid from its equation

$$y^2=\frac{x^3}{a-x}.$$

39. Trace the form of the Witch of Agnesi from its equation

$$x(a^2+y^2)=2a^3.$$

40. Solve the simultaneous equations:

(i) $4x^2-6xy+3=0$, $9y^2-6xy=4$;

(ii) $4xy=8y-7$, $2x^2+3y=2$;

(iii) $2x^2-xy=3$, $(y+2x-3)(y-x+1)=0$;

(iv) $xy=x+y$, $x^2+y^2-2x=2$;

(v) $xy=x+2y$, $x^2+y^2-2y=4$.

89. Freedom Equations. In this article we show how a curve may be drawn when its freedom equations are given.

I.
$$x=\frac{1-t^2}{1+t^2},\quad y=\frac{t}{1+t^2}.$$

First draw the graphs of $x=\dfrac{1-t^2}{1+t^2}$ and $y=\dfrac{t}{1+t^2}$. The graphs in Fig. 82 were drawn by the methods of this chapter according to the following table:

t	± 3	± 2	± 1	0	large	small
x	$-0{\cdot}8$	$-0{\cdot}6$	0	1	$-1+2/t^2$	$1-2t^2$
y	$\pm 0{\cdot}3$	$\pm 0{\cdot}4$	$\pm\frac{1}{2}$	0	$1/t$	$t-t^3$

A rough graph of y, considered as a function of x, can now be quickly drawn by observing from the above graphs how x and y simultaneously vary as t varies from $-\infty$ to $+\infty$. The graph in Fig. 82, showing the variation of y as x varies, was drawn by noting the general trend of the variation of y as x varies, and plotting certain chosen

points, shown in the diagram. The process is exhibited in the subjoined table:

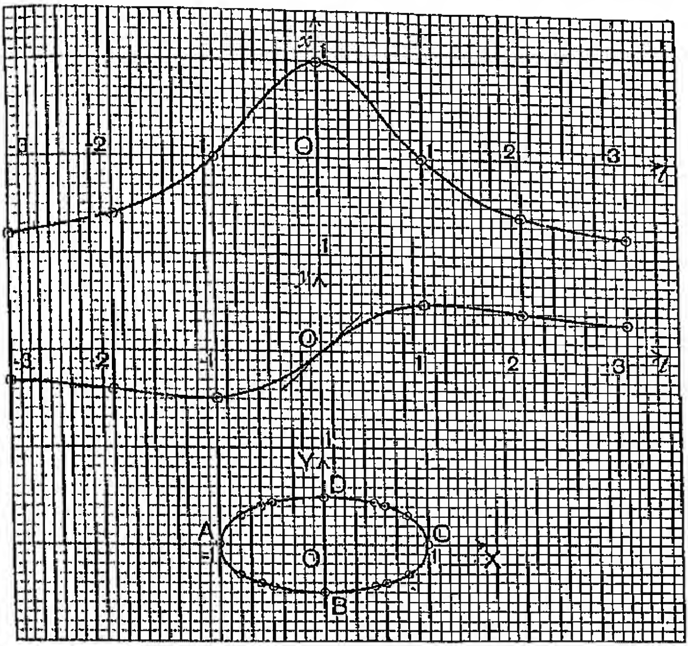


FIG. 82.

t	all but $-\infty$	inc. to -3	inc. to -2	inc. to $-1{\cdot}75$	inc. to -1
x	$-1+$	inc. to $-0{\cdot}8$	inc. to $-0{\cdot}6$	inc. to $-0{\cdot}51$	inc. to 0
y	$0-$	dec. to $-0{\cdot}3$	dec. to $-0{\cdot}4$	dec. to $-0{\cdot}43$	dec. to $-0{\cdot}5$

A to *B*

t	inc. to 0	inc. to 1	inc. to $+\infty$
x	inc. to 1	dec. to 0	dec. to -1
y	inc. to 0	inc. to $0{\cdot}5$	dec. to 0

B to *C*   *C* to *D*   *D* to *A*   in Fig. 82.

II. $$x=t(t-1), \quad y=t(t+1).$$

The graph of $x=t(t-1)$ is shown as a full line, and the graph of $y=t(t+1)$ as a dotted line in the diagram of Fig. 83. The following table shows the trend of the x, y graph:

t	$-\infty$	inc. to -1	inc. to $-\frac{1}{2}$	inc. to 0	inc. to $\frac{1}{2}$	inc. to 1	inc. to $+\infty$
x	$+\infty$	dec. to 2	dec. to $\frac{3}{4}$	dec. to 0	dec. to $-\frac{1}{4}$	inc. to 0	,, ,,
y	$+\infty$	dec. to 0	dec. to $-\frac{1}{4}$	inc. to 0	inc. to $\frac{3}{4}$	inc. to 2	,, ,,
		A to B	B to C	C to O	O to D	D to E	E to F

in Fig. 83.

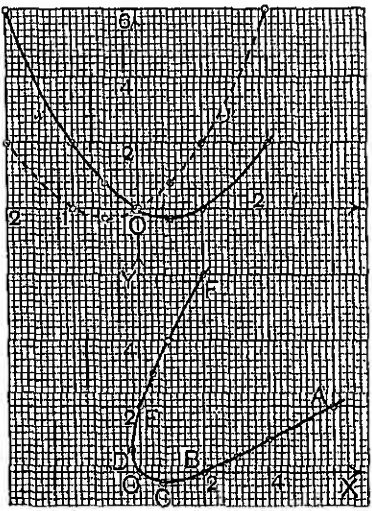


FIG. 83.

Ex. 1. Express the freedom equations
$$x=t(t-1), \quad y=t(t+1)$$
as a constraint equation in x, y.

By subtraction we have $y-x=2t$, and therefore $t=\frac{1}{2}(y-x)$. Substituting $\frac{1}{2}(y-x)$ for t in the first equation, we get
$$x=\frac{y-x}{2}\left(\frac{y-x}{2}-1\right) \quad \text{or} \quad x^2-2xy+y^2-2x-2y=0.$$

This process is called elimination, and we are said to have eliminated t between the two given equations.

Ex. 2. Find the points of intersection of the graphs specified by the equations
$$x=t(t-1),\quad y=t(t+1). \quad\text{.........(1)}$$
$$x=1+u,\quad y=4+2u. \quad\text{.........(2)}$$

We require that the equations
$$t(t-1)=1+u, \quad\text{.........(3)}$$
$$t(t+1)=4+2u, \quad\text{.........(4)}$$
should hold simultaneously.

From (3) and (4) $\quad 2t=3+u$ or $u=2t-3$.

Hence, from (3),
$$t(t-1)=2t-2,\quad i.e.\ t^2-3t+2=0,\quad i.e.\ t=1 \text{ or } 2.$$
Hence, from (1), $\quad x=0$ or 2, $\quad y=2$ or 6,
so that the points of intersection are $(0, 2)$ and $(2, 6)$.

Note. We might have expressed equations (1) and (2) as constraint equations, viz.
$$x^2-2xy+y^2-2x-2y=0, \quad\text{.........(1')}$$
$$2x-y+2=0, \quad\text{.........(2')}$$
and then solved (1′) and (2′) as simultaneous equations.

Freedom equations should be translated into constraint equations, if the latter can be readily found and handled.

Ex. 3. Prove that the constraint equation of the graph specified by
$$x=\frac{1-t^2}{1+t^2},\quad y=\frac{t}{1+t^2}$$
is
$$x^2+4y^2=1.$$

Ex. 4. Prove that the coordinates of the points of intersection of
$$x=\frac{1-t^2}{1+t^2},\quad y=\frac{t}{1+t^2}\quad\text{and}\quad x=-2+7u,\ y=\tfrac{1}{2}-\tfrac{1}{2}u$$
are $(0{\cdot}8, 0{\cdot}3)$ and $(-0{\cdot}6, 0{\cdot}4)$, and that the corresponding values of t are $\frac{1}{3}$, 2 and of u, $\frac{2}{5}$ and $\frac{1}{5}$.

Ex. 5. If $x=Vt\cos\alpha$, $y=Vt\sin\alpha-\frac{1}{2}gt^2$, where V, α, g are constants, draw the graphs of x and y considered as functions of t, and then draw the graph of y considered as a function of x. The values of t may be restricted to the range from $t=0$ to $t=2V\sin\alpha/g$. Find also the constraint equation of the curve.

EXERCISES XXIV.

1. Draw the graph showing the variation of y as x varies when
$$x=(t-1)(t-2),\quad y=t^2.$$
What is the constraint equation of the graph?

2. Draw to x, y axes the graph of

$$x=t(t^2-1),\quad y=t(t^2+1),$$

making the form near the origin quite clear. What is the constraint equation of the graph?

3. Trace the graphs of

$$\text{(i) } x=(t-1)(t-3),\quad y=t(t-2);$$

$$\text{(ii) } x=t(t-2),\quad y=t^2-1.$$

4. Trace the graph of

$$x=t^2,\quad y=\frac{t}{(t-1)^2}.$$

What is the constraint equation of the graph?

5. Trace the graph of

$$x=(t+1)^2,\quad y=t(t+1)^2,$$

and find the constraint equation.

6. Trace the graph of

$$x=\frac{t^2}{t-1},\quad y=\frac{t}{t^2-1},$$

and find the constraint equation.

7. Trace the graph of

$$x=\frac{3t}{1+t^3},\quad y=\frac{3t^2}{1+t^3},$$

and find the constraint equation.

8. Trace the graph of

$$x=\frac{t+1}{t-1},\quad y=\frac{2t}{t^3-1},$$

and find the constraint equation

CHAPTER XII.

IRRATIONAL FUNCTIONS.

90. Graph of $y=x-1\pm 2\sqrt{(x-2)}$. In §§ 83, 87 some equations of the form $y^2=f(x)$ were discussed; in this chapter we shall show how the general shape of curves given by equations of the form

$$y=ax+b\pm\sqrt{f(x)} \quad \text{or} \quad (y-ax-b)^2=f(x)$$

may be obtained when $f(x)$ is of a simple type. We begin with the equation

$$y=x-1\pm 2\sqrt{(x-2)}, \quad\text{.....................(1)}$$

which may be expressed in the forms

$$(y-x+1)^2=4(x-2),$$

$$x^2-2xy+y^2-6x+2y+9=0. \quad\text{...............(2)}$$

From equation (2), by arranging it in powers of y and solving the quadratic so obtained in terms of x, we derive equation (1).

First draw the graph of $y_1=x-1$, the line AG in Fig. 84.

For real values of y, the values of x must be equal to or greater than 2. Mark on the line AG the points $A, B, C, D, E, F, G, \ldots$, whose abscissae are 2, 3, 4, 5, 6, 7, 8, ...; then the table

x	2	3	4	5	6	7	8
$\pm 2\sqrt{(x-2)}$	± 0	± 2	$\pm 2{\cdot}83$	$\pm 3{\cdot}46$	± 4	$\pm 4{\cdot}47$	$\pm 4{\cdot}90$
	A	Bb Bb'	Cc Cc'	Dd Dd'	Ee Ee'	Ff Ff'	Gg Gg'

gives the points A, b and b', c and c', ... on the curve by the following rule. Bb and Bb', Cc and Cc', ... are steps whose measures are $+2$ and -2, $+2{\cdot}83$ and $-2{\cdot}83$, ..., these numbers being obtained from the second row and from the same column as contains the steps; thus, $Bb=+2$ and $Bb'=-2$. To plot the point b move 2 units upwards from

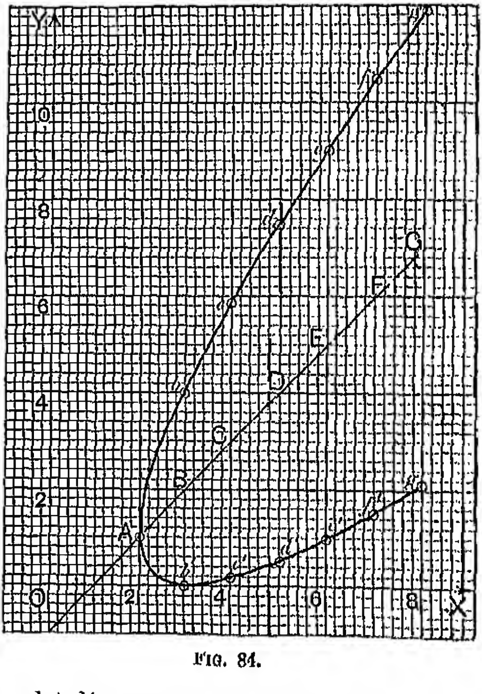


FIG. 84.

B, and to plot b' move 2 units downwards from B, and so on. The two steps from A are both zero, so that A itself is on the curve.

Note (i) that $x=2$ is the tangent at the point (2, 1) on the curve; (ii) that the line $y=x-1$ bisects all chords parallel to the y-axis.

The curve is a parabola and AG is a "diameter" of the parabola.

91. Graph of $y=\frac{x}{4}-1\pm\sqrt{\{(x-2)(x+1)\}}$. If y is real, the product $(x-2)(x+1)$ must be positive or zero. The graph of $(x-2)(x+1)$ is a festoon cutting $X'OX$, where $x=-1$ and where $x=2$, and therefore x cannot lie between -1 and 2. (A very rough sketch of the graph of $(x-2)(x+1)$ or a mental picture will show at once that the ordinate is negative for values of x between -1 and 2.)

Draw the graph of $y_1=\frac{x}{4}-1$, the line HD in Fig. 85. Mark on HD the points $A, B, C, D, \ldots E, F, G, H, \ldots$, whose abscissae are $2, 3, 4, 5, \ldots -1, -2, -3, -4, \ldots$. Then the table

x	2	3	4	5	...	-1	-2	-3	-4
$\pm\sqrt{(x-2)(x+1)}$	± 0	± 2	$\pm 3{\cdot}16$	$\pm 4{\cdot}24$	...	± 0	± 2	$\pm 3{\cdot}16$	$\pm 4{\cdot}24$
	A	Bb Bb'	Cc Cc'	Dd Dd'	... ...	Ee Ee'	Ff Ff'	Gg Gg'	Hh Hh'

gives the points A, b and b', c and c', ... on the curve (Fig. 85).

The equation may be written in the forms

$$\left(y-\frac{x}{4}+1\right)^2=(x-2)(x+1),$$

$$15x^2+8xy-16y^2-8x-32y-48=0\text{; } \quad \ldots\ldots\ldots(1)$$

from the second of these forms the given equation may be derived by arranging it as a quadratic in y and then solving for y in terms of x.

Note (i) that $x=-1$ is the tangent at the point $(-1, -\frac{5}{4})$ and $x=2$ the tangent at the point $(2, -\frac{1}{2})$; (ii) that $y=\frac{x}{4}-1$ bisects all chords of the curve parallel to the y-axis. The curve is a **hyperbola** and HD is a "diameter" of the hyperbola.

The expression on the left side of equation (1) may by a factorising process be written

$$(5x-4y-6)(3x+4y+2)-36\text{;}$$

hence the equation of the hyperbola may be expressed in the form

$$(5x-4y-6)(3x+4y+2)=36. \quad\text{..............}(2)$$

The asymptotes of the curve are the lines

$$5x-4y-6=0, \quad 3x+4y+2=0;$$

they are shown in the diagram.

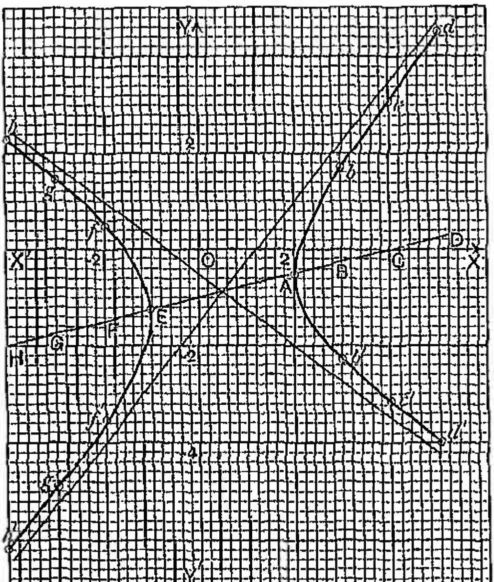


FIG. 85.

92. Graph of $y=\frac{x}{2}-1\pm\sqrt{\{(x+2)(6-x)\}}$. If y is real, the product $(x+2)(6-x)$ must be positive or zero. The graph of $(x+2)(6-x)$ is an inverted festoon cutting the x-axis at $x=-2$ and $x=6$, and therefore x must lie between -2 and 6.

Draw the graph of $y_1=\frac{x}{2}-1$, the line AK in Fig. 86.

Mark on AK the points $A, B, C, D, E, \ldots K$, whose abscissae are $-2, -1, 0, 1, 2, \ldots 6$. Then the table

x	-2	-1	0	1	2	3	4	5	6
$\pm\sqrt{(x+2)(6-x)}$	± 0	$\pm 2{\cdot}65$	$\pm 3{\cdot}46$	$\pm 3{\cdot}87$	± 4	$\pm 3{\cdot}87$	$\pm 3{\cdot}46$	$\pm 2{\cdot}65$	± 0
	A	Bb Bb'	Cc Cc'	Dd Dd'	Ee Ee'	Ff Ff'	Gg Gg'	Hh Hh'	K

gives the points A, b and b', c and c', ... on the curve.

The equation may be written in the forms

$$\left(y-\frac{x}{2}+1\right)^2=(x+2)(6-x),$$

$$5x^2-4xy+4y^2-20x+8y-44=0.$$

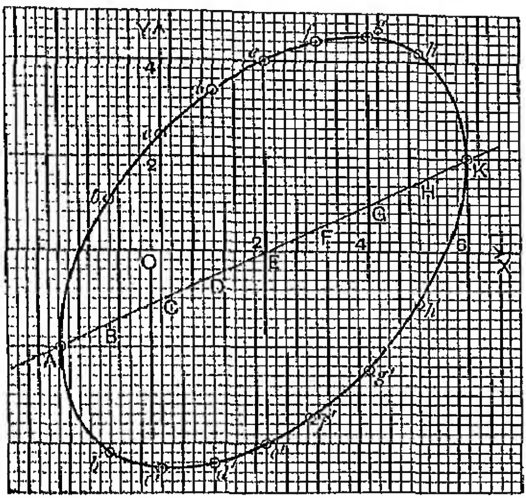


FIG. 86.

Note (i) that $x=-2$ is the tangent at the point $(-2, -2)$ and $x=6$ the tangent at the point $(6, 2)$; (ii) that $y=\frac{x}{2}-1$ bisects all chords of the curve parallel to the y-axis. The curve is an ellipse and AK is a "diameter" of the ellipse.

93. Graph of $y=\frac{x}{2}-1\pm\sqrt{(x^2+x+1)}$. If y is real, x^2+x+1 must be positive or zero. The graph of x^2+x+1 is festoon which does not cut the x-axis, and therefore y is real for every value of x.

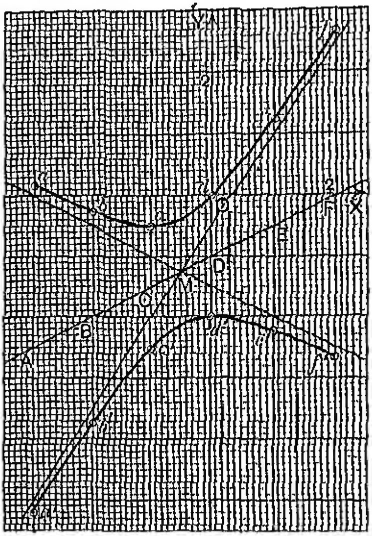


FIG. 87.

Draw the graph of $y_1=\frac{x}{2}-1$, the line AF in Fig. since y is real for every value of x the curve extends two branches from left to right, one above AF and other below AF.

Mark on AF the points A, B, C, D, E, F, whose abscissæ are $-3, -2, -1, 0, 1, 2$. Then the table

x	-3	-2	-1	0	1	2
$\pm\sqrt{(x^2+x+1)}$	$\pm 2{\cdot}65$	$\pm 1{\cdot}73$	± 1	± 1	$\pm 1{\cdot}73$	$\pm 2{\cdot}65$
	Aa	Bb	Cc	Dd	Ee	Ff
	Aa'	Bb'	Cc'	Dd'	Ee'	Ff'

gives the points a and a', b and b', ... on the curve.

The equation may be written in the forms

$$\left(y-\frac{x}{2}+1\right)^2=x^2+x+1,$$

$$3x^2+4xy-4y^2+8x-8y=0. \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Note that $y=\frac{x}{2}-1$ bisects all chords of the curve parallel to the y-axis. The curve is a **hyperbola** and AF is a "diameter" of the hyperbola.

Equation (1) may by a factorising process be expressed in the form

$$(3x-2y-1)(x+2y+3)=-3.$$

The asymptotes of the hyperbola are the lines

$$3x-2y-1=0, \quad x+2y-3=0;$$

these lines are shown in the diagram.

94. Graph of $y=x\pm\sqrt{\left(\frac{x-1}{x-2}\right)}$. If y is real, x must not lie between 1 and 2.

Draw the graphs of the two equations

$$y_1=x, \quad y_2=\pm\sqrt{\left(\frac{x-1}{x-2}\right)};$$

then

$$y=y_1+y_2,$$

and any ordinate of the curve is obtained by adding (algebraically) the corresponding ordinates y_1 and y_2. A sketch of the curve is shown in Fig. 88.

Some of the details of the graph were obtained as follows:

(1) x large. By Descending Continued Division

$$\frac{x-1}{x-2}=1+\frac{1}{x} \text{ approx.},$$

and by the ordinary process of extracting the square root

$$\sqrt{\left(1+\frac{1}{x}\right)}=1+\frac{1}{2x} \text{ approx.}$$

Hence, $$y=x\pm\sqrt{\left(\frac{x-1}{x-2}\right)}=x\pm\left(1+\frac{1}{2x}\right) \text{ approx.}$$

The lines $y=x+1$ and $y=x-1$ are asymptotes; the graph appears above $y=x+1$ to the far right and below to the far left, but appears below $y=x-1$ to the far right and above to the far left.

(2) $y=0$ when $0=x\pm\sqrt{\left(\frac{x-1}{x-2}\right)}$, that is, when

$$x^2(x-2)=x-1 \quad \text{or} \quad x^3-2x^2-x+1=0.$$

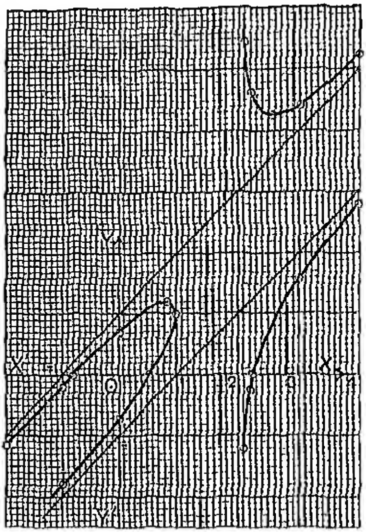


FIG. 88.

This equation may be solved graphically (see Chap. XV.); its roots are approximately $-0{\cdot}80$, $0{\cdot}55$, $2{\cdot}25$. These give the points where the curve crosses the x-axis.

(3) The following table gives a number of points:

x	-2	-1	0	1	2	$2{\cdot}1$	$2{\cdot}2$	3	4
y	$-1{\cdot}13$ $-2{\cdot}87$	$-0{\cdot}18$ $-1{\cdot}82$	$\pm 0{\cdot}71$	1	$+\infty$ $-\infty$	$5{\cdot}42$ $-1{\cdot}22$	$4{\cdot}65$ $-0{\cdot}25$	$4{\cdot}41$ $1{\cdot}59$	$5{\cdot}22$ $2{\cdot}78$

EXERCISES XXV.

1. Graph the equations:

(i) $y=x\pm\sqrt{x}$; (ii) $y=\frac{x}{2}\pm\sqrt{(x+1)}$; (iii) $(4y-x)^2=16(x-2)$;
(iv) $4x^2-12xy+9y^2=4x$; (v) $x^2-2xy+y^2-2x+4y-3=0$.

2. Graph the equation $y=x-1\pm2\sqrt{\{(x-2)(x+1)\}}$. What are the asymptotes of the curve?

3. Graph the equation $y=x-1\pm2\sqrt{\{(x+2)(6-x)\}}$.
Find the equation (i) of the diameter that bisects chords parallel to the y-axis; (ii) of the diameter that bisects chords parallel to the x-axis.

4. Graph the equation

$$8x^2+6xy-9y^2+4x-12y-13=0.$$

Find (i) the equation of the diameter that bisects chords parallel to the y-axis; (ii) the equation of the diameter that bisects chords parallel to the x-axis; (iii) the equations of the asymptotes.

5. Graph the equation

$$10x^2-6xy+9y^2+2x-6y-8=0.$$

Find the equation of the diameter that bisects chords (i) parallel to the y-axis; (ii) parallel to the x-axis.
For what values of y are the values of x equal, and what is the nature of the corresponding points on the curve?

6. What values of y give equal values of x in the following equations?

(i) $3x^2+4xy-4y^2+8x-8y=0$;
(ii) $5x^2-4xy+4y^2-20x+8y-44=0$.

Graph the equations.

7. Draw the curves given by the following equations:

(i) $4x^2+9y^2=36$; (ii) $4x^2-9y^2=36$; (iii) $4x^2-9y^2=-36$;
(iv) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; (v) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$; (vi) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$.

8. Simplify the following equations by change of axes (§§ 48, 49), and then graph the equations:

(i) $x^2+2x+2y^2-4y=13$; (ii) $x^2+2x-2y^2+4y=17$;
(iii) $4\left(\frac{x+y}{\sqrt{2}}\right)^2+9\left(\frac{x-y}{\sqrt{2}}\right)^2=36$; (iv) $4(x+y)^2-9(x-y)^2=72$;
(v) $4(x+y)^2-9(x-y)^2=-72$; (vi) $\frac{(3x-4y+1)^2}{225}+\frac{(4x+3y)^2}{100}=1$;
(vii) $(x-2y+1)^2+(2x+y+1)^2=5$;
(viii) $4(x-2y+1)^2+(2x+y+1)^2=20$;
(ix) $(x-2y+1)^2-4(2x+y+1)^2=20$;
(x) $(x-2y+1)^2-4(2x+y+1)^2=-20$.

9. Graph the following equations:

(i) $(y-x)^2=x^3$; (ii) $(y-x)^3=x^5$; (iii) $xy^2=(x-1)(x-2)$;

(iv) $y=x\pm\sqrt{\{(x-1)(x-2)(x-3)\}}$; (v) $y=x\pm2\sqrt{\left(\dfrac{x-1}{x^2+1}\right)}$;

(vi) $y=x^2\pm\sqrt{x}$; (vii) $y=x^2\pm\sqrt{x^3}$; (viii) $y=x^2\pm\sqrt{(x-x^2)}$.

10. Solve the following simultaneous equations, and verify the solutions by graphs:

$$\text{(i)}\ \begin{cases}2(x^2+y^2)-3xy=3x+2y,\\3(x^2+y^2)-4xy=5x+3y.\end{cases}$$

$$\text{(ii)}\ \begin{cases}x^2-5xy+2y^2=2x-4y,\\2x^2-3xy+2y^2=4x.\end{cases}$$

$$\text{(iii)}\ \begin{cases}x^2+xy+y^2=(a+b)(x+y),\\x^2-xy+y^2=(a-b)(x-y).\end{cases}$$

CHAPTER XIII.

SUCCESSIVE APPROXIMATIONS.

95. Change of Origin. If $\xi'\omega\xi$, $\eta'\omega\eta$ are rectangular axes, parallel respectively to $X'OX$, $Y'OY$, it has been shown (§ 48) that the coordinates x, y of any point P, referred to the axes through O, are connected with the coordinates ξ, η of P, referred to the axes through ω, by the equations

$$x=h+\xi,\ y=k+\eta\ \ldots(1),\quad \xi=x-h,\ \eta=y-k\ \ldots(2);$$

where h, k are the coordinates of the new origin ω with reference to the axes through O. An equation can often be simplified by change of origin, and we shall show in this chapter how a change of origin enables us to obtain more accurate graphs with comparatively little labour. The following examples should be carefully noted.

Ex. 1. Show that the equation $y=ax^2+bx+c$ may by change of origin be reduced to the form $\eta=a\xi^2$.

By the process of "completing the square," we find that the given equation may be put in the form

$$y=a\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a}\quad\text{or}\quad y+\frac{b^2-4ac}{4a}=a\left(x+\frac{b}{2a}\right)^2.$$

Now let $x+\frac{b}{2a}=\xi$, $y+\frac{b^2-4ac}{4a}=\eta$, and we get $\eta=a\xi^2$. If we write the equations of transformation in the form

$$x=-\frac{b}{2a}+\xi,\quad y=-\frac{b^2-4ac}{4a}+\eta,$$

and compare with equations (1), we see that the coordinates h, k of the new origin are

$$h=-\frac{b}{2a},\quad k=-\frac{b^2-4ac}{4a}.\ \ldots\ldots\ldots\ldots(3)$$

The graph of $\eta=a\xi^2$ is clearly *the same curve* as the graph of $y=ax^2$ if both curves are plotted to the same scales. Hence the graph of $y=ax^2+bx+c$ is identical with the graph of $y=ax^2$ except that it occupies a different position with respect to the axes $X'OX$, $Y'OY$. The shape of the curve therefore depends solely on the constant a.

The graph of $y=ax^2$ is a parabola (§ 77), and the y-axis is called the axis of the parabola. When a is positive the origin is the lowest point, and when a is negative the highest point of the curve; this point is called the vertex of the parabola. The graph of $y=ax^2+bx+c$ is therefore a parabola, and when the equation has been reduced to the form $\eta=a\xi^2$ the new origin is the vertex; equations (3) give the coordinates of the vertex referred to the axes $X'OX$, $Y'OY$.

When a is positive the curve is often called a "festoon," and when a is negative an "inverted festoon." It should be thoroughly fixed in the student's memory that the curve is a festoon or an inverted festoon according as a is positive or negative.

Ex. 2. Show that $(x-1)(x-2)$ is positive or negative according as x does not or does lie between 1 and 2.

The graph of $(x-1)(x-2)$ is a festoon which cuts the x-axis where $x=1$ and where $x=2$; and therefore the ordinate is negative between these two points, but positive if the x of the point is not between 1 and 2.

Ex. 3. Show that $a(x-1)(x-2)$ is negative when x lies between 1 and 2 and a is positive, but positive for the same range of x when a is negative.

If a is positive the curve is a festoon, but if a is negative the curve is an inverted festoon, and the results follow as in Example 2.

Ex. 4. If the roots α, β of the equation $ax^2+bx+c=0$ are real and unequal, show that, when a is positive, the expression ax^2+bx+c is positive or negative according as x does not or does lie between α and β, but when a is negative the expression is positive or negative according as x does or does not lie between α and β.

When a is positive, the curve is a festoon which cuts the x-axis where $x=\alpha$ and where $x=\beta$; when a is negative, the curve is an inverted festoon which cuts the x-axis at the same two points.

Ex. 5. Show that x^2+x+1 is positive for every real value of x.

The roots of $x^2+x+1=0$ are imaginary; therefore the graph of x^2+x+1 does not cross or meet the x-axis. Hence the ordinate has always the same sign; when $x=0$ the ordinate $=1$, so that the ordinate is always positive.

Ex. 6. Show that if the roots of the equation $ax^2+bx+c=0$ are imaginary, the expression ax^2+bx+c has always the same sign as c or a.

The graph does not cross the x-axis, so that the ordinate has always the same sign; to find the sign, put $x=0$. When the roots are imaginary, a and c have the same sign.

Ex. 7. If a, b, c are in ascending order of magnitude, prove that one of the roots of the equation

$$\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}=0$$

lies between a and b, and the other between b and c.

Ex. 8. Find the necessary and sufficient conditions that $ax^2+2bx+c$ should have (i) the positive sign, (ii) the negative sign, whatever the value of x.

96. Shape of a Graph near a given Point on it. In this and following articles we shall show how to obtain rapidly the shape of a graph in the neighbourhood of any given point on it; we begin with the graph of a quadratic function.

Let it be required to examine the shape of the graph of

$$y=x^2-x$$

near the origin, which is obviously a point on the graph.

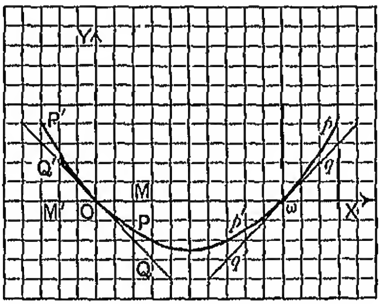


FIG. 89.

We must examine the relative importance of the terms x^2 and $-x$ when x is a small positive or negative fraction. Now when x is small, say $1/10^3$, x^2 is $1/10^6$, which is much smaller; the error in taking $-x$ instead of x^2-x as the value of y is $1/10^6$, or just a little more than 0·1 per cent. of the true value of y. When x is $1/10^4$ the error in taking $-x$ instead of x^2-x is $1/10^8$, or about 0·01 per cent. of the true value; and so on. Now draw the straight line $Q'OQ$ (Fig. 89), which is the graph of $y=-x$.

Near the origin the graph of $y=x^2-x$ must lie close to $Q'OQ$, but does not coincide with $Q'OQ$; the only point these two graphs have in common is O. Further,

$$x^2-x=-x+x^2=-x+(\text{positive quantity}),$$

whether x is positive or negative. Hence $P'OP$, the graph of $y=x^2-x$ near O, is derived from $Q'OQ$, the graph of $y=-x$, by drawing through O a curved line close to $Q'OQ$ and *above* it on *both* sides of O, as in Fig. 89. In the figure, if $OM=x$, $MQ=-x$, $QP=x^2$, QP being therefore a *positive* step, then (§ 3)

$$MP=MQ+QP=-x+x^2;$$

if $OM'=x$, $M'Q'=-x$, $Q'P'=x^2$, $Q'P'$ being like QP a positive step, then

$$M'P'=M'Q'+Q'P'=-x+x^2.$$

The equation $y=-x$ is the closest approximation of the first degree to the equation $y=-x+x^2$, where x is small, and is called the **first approximation** near the origin. For the sake of distinction, the y of the first approximation is often called y_1, so that $y_1=-x$ is then written as the first approximation to $y=-x+x^2$.

The Tangent at the Origin. Of all straight lines which pass through O, the line $y=-x$, which is the graph of the first approximation to $y=-x+x^2$, lies closest to the curve; this line is the tangent at O to the curve.

We thus see that near O the curve is a part of a festoon; the curve is **concave upwards.**

If we want now to examine the shape near any other point, say the point $\omega(1, 0)$, we shift the origin to this point by putting $1+\xi$ for x and η for y, and then find the first approximation near the new origin. We thus have

$$\eta=(1+\xi)^2-(1+\xi)=\xi+\xi^2.$$

The first approximation is $\eta=\xi$, and the curve $p'\omega p$ lies, near ω, above the straight line $q'\omega q$, which is the graph of $\eta=\xi$.

The equation of the tangent at ω, referred to the new axes, is $\eta=\xi$; to find the equation referred to the old axes, we must put $x-1$ for ξ and y for η. Thus the equation of the tangent at $(1, 0)$ to the graph of $y=x^2-x$ is $y=x-1$.

It is easy now to form a mental picture of the shape near the origin of the graph of the equation

$$y = ax^2 + bx.$$

The first approximation, which represents the tangent at the origin, is $y_1 = bx$; the tangent has a right-hand upward slope if b is positive (like $q'\omega q$), but a right-hand downward slope if b is negative (like $Q'OQ$). If a is positive the curve lies *above* the tangent, but if a is negative the curve lies *below* the tangent.

To make himself quite familiar with the shape, the student should draw the graph for different positive and negative values of a and b, for example for the values ± 1, ± 2, ± 3 of a and b.

Ex. 1. Draw the graph of $y = x^2 - x$ near the point (3, 6), and show that the equation of the tangent at the point is

$$y = 5x - 9.$$

Putting $3 + \xi$ for x and $6 + \eta$ for y, we shift the origin to the point (3, 6). The equation becomes

$$6 + \eta = (\xi + 3)^2 - (\xi + 3) \quad \text{or} \quad \eta = 5\xi + \xi^2.$$

The tangent at the new origin is $\eta = 5\xi$, and the curve lies above the tangent near the new origin. To return to the old axes, put $x - 3$ for ξ and $y - 6$ for η; we thus get

$$y - 6 = 5(x - 3) \quad \text{or} \quad y = 5x - 9$$

as the equation of the tangent at (3, 6).

Ex. 2. Draw the graphs of the following equations near the points indicated, and find the equations of the tangents at these points:

(i) $y = x^2 - x$; point (2, 2); (ii) $y = (x-2)(x-3)$; points (2, 0), (3, 0);
(iii) $y = (x-1)(x-2)$; points (3, 2), $(-1, 6)$;
(iv) $y = 2x^2 - 3x + 1$; point (2, 3);
(v) $y = (x-1)(x-2)(x-3)$; points (1, 0), (3, 0).

97. Point of Inflexion. Consider the graph of the equation

$$y = x^3 - x.$$

Near the origin the first approximation is $y = -x$, represented by $Q'OQ$ of Fig. 90. The graph of $y = x^3 - x$ near O lies close to $Q'OQ$.

When x is positive, we have

$$y = -x + x^3 = -x + (\text{positive quantity});$$

but when x is negative we have, since x^3 is now negative,

$$y=-x+x^3=-x+(\text{negative quantity})$$
$$=-x-(\text{positive quantity}).$$

Hence $P'OP$, the graph of x^3-x near the origin, is derived from $Q'OQ$ by drawing a curved line close to $Q'OQ$, *above* it to the right of O and *below* it to the left as in

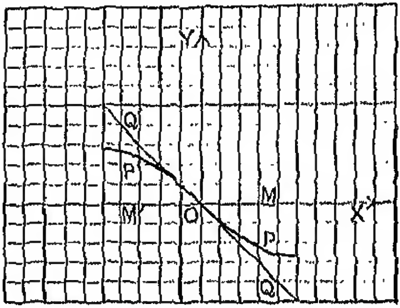


FIG. 90.

Fig. 90. In the figure, if $OM=x$, $MQ=-x$, $QP=x^3$, QP being a positive step since x is positive, then

$$MP=MQ+QP=-x+x^3;$$

if $OM'=x$, $M'Q'=-x$, $Q'P'=x^3$, $Q'P'$ being now a negative step since x is negative, then

$$M'P'=M'Q'+Q'P'=-x+x^3.$$

$P'OP$ touches $Q'OQ$ at O and also *crosses* $Q'OQ$ at O. The origin is a **Point of Inflexion** on the graph of $y=x^3-x$, and the tangent $y=-x$ is an **inflexional tangent** (§ 77).

The student should now work the following **examples** so as to recognise at once the shape of a curve near a **point** of inflexion.

Ex. 1. Draw the graphs near the origin of

(i) $y=x^3-2x$; (ii) $y=x^3+x$; (iii) $y=x^3+2x$;
(iv) $y=x-x^3$; (v) $y=2x-x^3$; (vi) $y=-x-x^3$;
(vii) $y=ax^3+bx$.

Ex. 2. Draw the graph of $y=(x-1)(x-2)(x-3)$ near the point $(2, 0)$.

Ex. 3. Show that (1, 4) is a point of inflexion on the graph of $y=x^3-3x^2+5x+1$, and that $y=2x+2$ is the equation of the inflexional tangent.

Ex. 4. Draw the graph of $y=x-x^5$ near the origin.

98. The Polynomial for Small Values of x. Let the polynomial be written in ascending powers of x; the equation is thus of the form

$$y=a+bx+cx^2+dx^3+ex^4+fx^5+\dots\dots\dots\dots\dots(1)$$

Take a definite example, say

$$y=5+2x-3x^2+4x^3+2x^4. \quad \dots\dots\dots\dots\dots(2)$$

When $x=0$, $y=5$; without actually shifting the origin to the point (0, 5), we can see that the first approximation* to equation (2) when x is small is

1st app. $\qquad y_1=5+2x. \quad \dots\dots\dots\dots\dots(3)$

The error in this approximation is $-3x^2+4x^3+2x^4$; the ratio of the error to the term in x that has been retained is

$$\frac{-3x^2+4x^3+2x^4}{2x}, \text{ that is, } -\frac{3}{2}x+2x^2+x^3.$$

The error is therefore small compared with $2x$ when x is small.

To find a second approximation we retain the term $-3x^2$, and write, denoting by y_2 the second approximation,

2nd app. $\qquad y_2=5+2x-3x^2=y_1-3x^2. \quad \dots\dots\dots\dots\dots(4)$

The error in this approximation is $4x^3+2x^4$; the ratio of the error to the last term retained, namely $-3x^2$, is

$$\frac{4x^3+2x^4}{-3x^2}, \text{ that is, } -\frac{4}{3}x-\frac{2}{3}x^2.$$

The error is therefore small compared with $-3x^2$ when x is small.

* For small values of x the value of y is nearly 5, and we might therefore say that $y=5$ is the first approximation; but for our purposes we need approximations that contain x. If we shift the origin to (0, 5) by putting ξ for x and $5+\eta$ for y, the first approximation at the new origin is obviously $\eta=2\xi$. Going back to the old origin by putting x for ξ and $y-5$ for η, we get $y-5=2x$ or $y=5+2x$, which is equation (3).

We can now see what the shape of the curve is near the point (0, 5); see Fig. 91. A is the point (0, 5); BAC is the graph of the first approximation and is the tangent at A; DAE is the graph of the second approximation which is part of an inverted festoon and is convex upwards, lying below BAC on both sides of A.

Close to A the graph of (2) cannot differ much from DAE; the difference EC, for example, between the ordinates of C and E, when x is the abscissa of each point, is $3x^2$ and, as we have seen, the sum of the terms in (2) that follow $-3x^2$ is small compared with $3x^2$ when x is small. The corresponding point on the graph of (4) must therefore lie below C close to E.

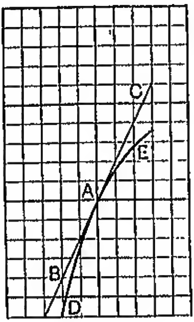


FIG. 91.

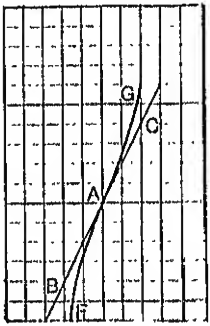


FIG. 92.

Consider now a case in which equation (1) contains no term in x^2, but does contain a term in x^3; that is, $c=0$, $d\neq 0$. Take for instance

$$y=5+2x+4x^3+2x^4. \qquad (5)$$

Near (0, 5) the first approximation is the same as before but there is now no term in x^2 to give us an approximation of the second degree in x—no second approximation. The graph of (5) must therefore, near A, lie much closer to BAC than DAE does. Denoting by y_3 the approximation to (5), which contains the term in x^3, and calling this the third approximation, we have

3rd app. $$y_3=5+2x+4x^3=y_1+4x^3. \qquad (6)$$

The error is here $2x^4$, and is small compared with $4x^3$ when x is small.

In this case A is a point of inflexion (Fig. 92), and BAC is an inflexional tangent.

If in (1) $c=0$, $d=0$, $e \neq 0$, then the graph of (1) near A resembles DAE (Fig. 91), but it lies much closer to BAC than DAE does. If $c=0$, $d=0$, $e=0$, $f \neq 0$, the graph near A resembles FAG (Fig. 92), but it lies much closer to BAC, which is an inflexional tangent.

Ex. Examine the shape of the curves given by the following equations near the points where they cross the y-axis.

(i) $y=7-4x+x^2-2x^4$; (ii) $y=7-4x+x^4$;

(iii) $y=7-4x+x^3-2x^4$; (iv) $y=7-4x+x^5$.

99. The Rational Fraction for Small Values of x. We shall now consider a fractional function, say

$$y=\frac{2x^2-x+3}{x^2+x+1}. \quad \text{.......................(1)}$$

First arrange numerator and denominator in ascending powers of x and divide, continuing the division as far, say, as the term of the quotient in x^3. The integral quotient is

$$3-4x+3x^2+x^3,$$

and the remainder is $-(4x^4+x^5)$, so that

$$y=3-4x+3x^2+x^3-\frac{4x^4+x^5}{1+x+x^2}$$
$$=3-4x+3x^2+x^3-R,$$

where $\quad R=(4x^4+x^5)/(1+x+x^2)$.

When $x=0$, $y=3$, and near the point $(0, 3)$ on the graph of (1) x is small. From the value of R, we have

$$\frac{R}{x^3}=\frac{4x+x^2}{1+x+x^2}.$$

When x is small, the numerator of the fraction is small, and the denominator differs but little from unity; hence when x is small, *R is small compared with x^3.* Proceeding as before, we find approximations.

1st app. $\qquad y_1 = 3 - 4x. \qquad (Q'AQ, \text{ Fig. } 93)$

2nd app. $\qquad y_2 = 3 - 4x + 3x^2$

$\qquad\qquad = y_1 + 3x^2. \qquad (P'AP, \text{ Fig. } 93)$

3rd app. $\qquad y_3 = 3 - 4x + 3x^2 + x^3$

$\qquad\qquad = y_2 + x^3.$

The 3rd approximation is not shown in the diagram; it could hardly be distinguished from the second.

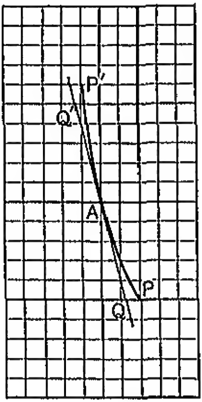


FIG. 93.

Thus, near the point (0, 3) the graph of (1) is a curved line which touches the line $y_1 = 3 - 4x$ and lies above it on both sides of the point.

It is easy now to examine the shape of a curve near any given point on it; if the point is (h, k), we have the rule.

Rule. *Shift the origin to the point (h, k) by putting $h + \xi$ for x and $k + \eta$ for y; then examine the shape of the curve, now given by the new equation, near the new origin by the above method.*

In next section we consider the form of a graph for large values of x.

Ex. 1. Use Ascending Continued Division to find approximations to the following equations for small values of x, and graph the equations for such values.

(i) $y = \dfrac{x}{x-1}$; (ii) $y = \dfrac{x^2}{x-1}$; (iii) $y = \dfrac{x-1}{(x-2)(x-3)}$;

(iv) $y = \dfrac{x^2 - x + 1}{x^2 + x + 1}$; (v) $y = \dfrac{x^3 - 2x + 1}{x^3 - x - 1}$; (vi) $y = \dfrac{x^4 + 4}{x^2 - x - 2}$;

(vii) $y = \dfrac{x^3 - 2}{x - 1}$.

Ex. 2. Find the equation of the tangent to the graph of $y = \dfrac{x-1}{x-2}$ at the point (3, 2).

Ex. 3. Prove that the part of any tangent to the graph of $y = \dfrac{1}{x}$ intercepted between the axes is bisected at the point of contact.

Show that the equation of the tangent at (h, k) is

$$y-k=-(x-h)/h^2,$$

which, since $hk=1$, may be written

$$\frac{x}{h}+\frac{y}{k}=2.$$

100. Graph of a Rational Function for Large Values of x. We apply the method of Descending Continued Division stated in Note (ii), p. 206; our chief purpose is to see how the curve appears in relation to its asymptotes for large values of x.

(A) Take first the equation $y=\dfrac{x^2+x+1}{x}$, which may be written

$$y=x+1+\frac{1}{x}.$$

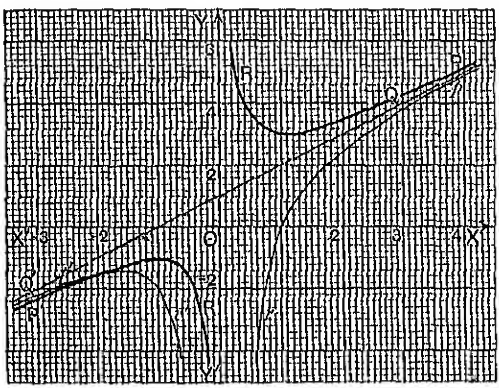


Fig. 94.

When x is large, we take as the first approximation*

$$y_1=x+1,$$

which is represented by $Q'Q$ in Fig. 94; $Q'Q$ is an asymptote.

*We take $y_1=x+1$ and not $y_1=x$, because when x tends to infinity y tends to $x+1$. For large values of x the curve runs close to the line given by $y=x+1$, while it remains at a finite distance from the line given by $y=x$.

When x is positive, $y=y_1+$(positive quantity); thus the curve appears at P, on the far right, *above* Q.

When x is negative, $y=y_1+$(negative quantity); thus the curve appears at P', on the far left, *below* Q'.

Ex. Show how the graphs of the following equations approach the asymptote $y=x+1$.

(i) $y=\dfrac{x^2+x-1}{x}$; (ii) $y=\dfrac{x^4+x^3+1}{x^3}$; (iii) $y=\dfrac{x^4+x^3-1}{x^3}$.

(B) Take next the equation $y=\dfrac{x^3+x^2-2}{x^2}$, which may be written

$$y=x+1-\frac{2}{x^2}.$$

When x is large, we take as the first approximation

$$y_1=x+1,$$

represented by $Q'Q$ of Fig. 94; $Q'Q$ is an asymptote.

In this case, whether x is positive or negative $2/x^2$ is positive, and therefore $y=y_1-$(positive quantity); hence the curve appears at p, on the far right, *below* Q and also at p', on the far left, *below* Q'.

Ex. Show how the graphs of the following equations approach the asymptote $y=x+1$.

(i) $y=\dfrac{x^3+x^2+2}{x^2}$; (ii) $y=\dfrac{x^5+x^4+2}{x^4}$; (iii) $y=\dfrac{x^5+x^4-2}{x^4}$.

(C) We shall take finally the equation

$$y=\frac{x^3-3x+2}{x^2-x+1}.$$

Apply the method of descending continued division, carrying the operation till the quotient contains at least one term with x in the denominator (that is, a term in $1/x$); we find

$$y=x+1-\frac{3}{x}-\frac{2-\dfrac{3}{x}}{x^2-x+1}=x+1-\frac{3}{x}-R,$$

where
$$R=\frac{2-\dfrac{3}{x}}{x^2-x+1}=\frac{\dfrac{1}{x^2}\left(2-\dfrac{3}{x}\right)}{1-\dfrac{1}{x}+\dfrac{1}{x^2}}.$$

We need to compare R with $-\frac{3}{x}$, which is the smallest term of the quotient $x+1-\frac{3}{x}$ when x is large. Now

$$\frac{R}{\frac{3}{x}}=\frac{\frac{1}{x}\left(2-\frac{3}{x}\right)}{3\left(1-\frac{1}{x}+\frac{1}{x^2}\right)}.$$

The numerator of this fraction is small when x is large, and the denominator differs but little from 3 for large values of x; hence, when x is large R is small compared with $3/x$.

Thus, when x is large, the first approximation is

$$y_1=x+1,$$

represented by $Q'Q$ of Fig. 95; $Q'Q$ is an asymptote.

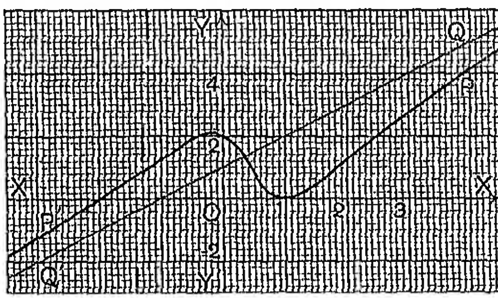


Fig. 95.

The second approximation is

$$y_2=x+1-\frac{3}{x}=y_1-\frac{3}{x}.$$

When x is large and positive, $y_2=y_1-$(positive quantity), but when x is (numerically) large and negative, $y_2=y_1$ +(positive quantity). Hence the curve appears at P, on the far right, below Q, and at P', on the far left, above Q'.

The graph is shown in Fig. 95. It will be a good exercise for the student to verify the form of the curve by examining the shape at such points as (1, 0), (0, 2), (–2, 0), and by plotting a number of points on the graph.

101. Worked Examples. We shall now work two examples to show how by applying the methods just indicated we may obtain fairly accurate graphs.

Ex. 1. $y=x(x-1)(x-2)$.

(i) Near the origin. $y=x^3-3x^2+2x$.

1st app. $y_1=2x$; draw the line BOC in Fig. 96.

2nd app. $y_2=2x-3x^2=y_1-3x^2$; draw the graph touching BOC at O and lying below BOC on both sides of O.

(ii) Near the point (1, 0). Put $x=\xi+1$, $y=\eta$.

Then $\eta=(\xi+1)\xi(\xi-1)=\xi^3-\xi$.

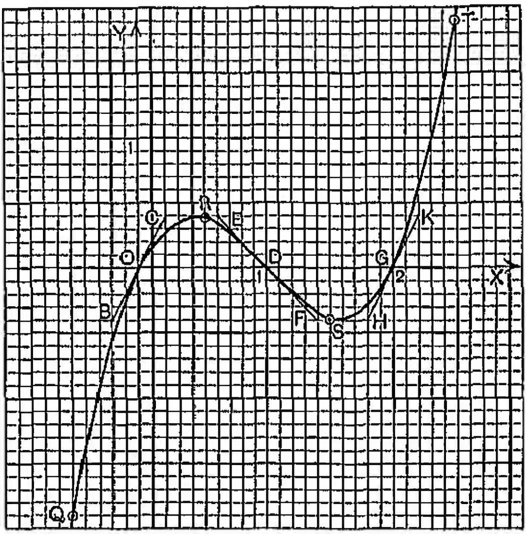


FIG. 96.

1st app. $\eta_1=-\xi$; draw the line EDF in Fig. 96.

2nd app. $\eta_2=-\xi+\xi^3=\eta_1+\xi^3$; draw the graph touching EDF at D and lying above it to the right of D and below it to the left.

(iii) Near the point (2, 0). Put $x=\xi+2$, $y=\eta$.

Then $\eta=(\xi+2)(\xi+1)\xi=\xi^3+3\xi^2+2\xi$.

1st app. $\eta_1=2\xi$; draw the line HGK.

2nd app. $\eta_2=2\xi+3\xi^2$; draw the graph touching HGK at G and lying above it on both sides of G.

(iv) Plot the points

$$Q(-0{\cdot}5,\ -1{\cdot}9),\quad R(0{\cdot}5,\ 0{\cdot}4),\quad S(1{\cdot}5,\ -0{\cdot}4),\quad T(2{\cdot}5,\ 1{\cdot}9),$$

and complete the graph in the usual way.

Ex. 2.
$$y=\frac{x(x-1)}{x-2}.$$

By Ascending Division, we find $y=\frac{1}{2}x-\frac{1}{4}x^2-\frac{1}{8}x^3-\ldots$.

(i) Near the origin.

1st app. $y_1=\frac{x}{2}$; draw BOC in Fig. 97.

2nd app. $y_2=\frac{x}{2}-\frac{x^2}{4}$; draw the graph touching BOC at O and lying below it on both sides of O.

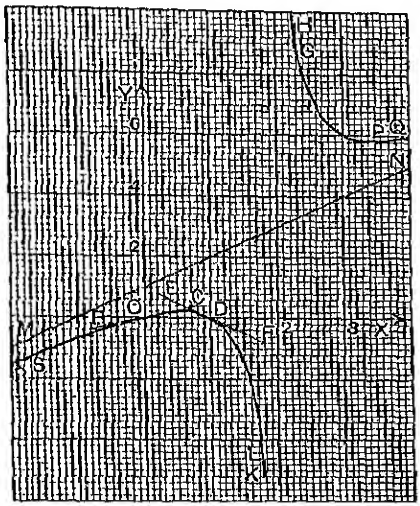


FIG. 97.

(ii) Near the point (1, 0). Put $x=\xi+1$, $y=\eta$; then

$$\eta=\frac{(\xi+1)\xi}{\xi-1}=-\xi-2\xi^2-2\xi^3-\ldots.$$

1st app. $\eta_1=-\xi$; draw EDF.

2nd app. $\eta_2=-\xi-2\xi^2$; draw the graph touching EDF at D and lying below it on both sides of D.

(iii) Near $x=2$. Put $x=\xi+2$, $y=\eta$; then

$$\eta=\frac{(\xi+2)(\xi+1)}{\xi}=\frac{2}{\xi}+3+\xi.$$

1st app. $\eta_1=\frac{2}{\xi}$; mark GH and KL in the diagram.

(iv) When x is large. By Descending Division, we find

$$y=x+1+\frac{2}{x}+\ldots.$$

1st app. $y_1=x+1$; draw MN.

2nd app. $y_2=x+1+\frac{2}{x}=y_1+\frac{2}{x}$; mark PQ and RS in the diagram.

(v) Plot guiding points, and complete the graph in the usual way.

EXERCISES XXVI.

1. Draw the graphs of the following equations near the origin:

(i) $y=2x-x^2$; (ii) $y=-x^2-x$; (iii) $y=x^3$;

(iv) $y=x+x^3$; (v) $2y=3x-4x^3$.

2. Draw the graph of

(i) $y=2+x+x^2$ near $(0, 2)$; (ii) $y=2+x+x^3$ near $(0, 2)$;

(iii) $y=2-x-x^3$ near $(0, 2)$; (iv) $y=2x-x^4$ near $(0, 0)$.

3. Use the methods of change of origin and successive approximation to draw the graphs of the following equations, plotting selected points to give the graph precision:

(i) $y=(x-1)(x-2)(x-3)$; (ii) $y=x^2(x-1)^2$;

(iii) $y=x(x-1)^3$; (iv) $y=x^2(x-1)(x+2)$;

(v) $y=x^3(x-1)$; (vi) $y=x(1+x^2)(1+x)$;

(vii) $y=(x+1)(x+2)^3$; (viii) $y=(x^2-1)(x^2-4)$.

4. Find the equations of the tangents to the curve $y=x^2-3x+2$ at the points $(-1, 6)$, $(0, 2)$, $(2, 0)$, $(3, 2)$. Draw these tangents and make a graph of the equation between $x=-1$ and $x=3$.

5. Draw the graph of $y=-2x^2+3x-2$ by examining the form of the graph near chosen points.

6. Prove that $y=x-1+\frac{2}{x}$ gives the approximate form of $y=\frac{x(x+1)}{x+2}$ when x is large. Draw the graph of $y=\frac{x(x+1)}{x+2}$.

7. Prove that $y=x/4+x^2/4$ gives the approximate form of $y=\frac{x(x+1)^2}{(x+2)^2}$ when x is small, and $y=x-2+\frac{5}{x}$, the approximate form when x is large. Draw the graph of $\frac{x(x+1)^2}{(x+2)^2}$.

8. Draw the graph of $y=(2x^2-x+3)/(x^2+x+1)$ when x is small, and also when x is large; complete the graph by plotting a number of points. Prove that the straight line $y=3-4x$ touches the curve at $(0, 3)$ and cuts the curve again at $(-3/4, 6)$.

9. Find the equation of the tangent to the curve $y=\frac{x^3+x+2}{x^2-x+2}$ at the point $(1, 2)$.

Shift the origin to the point $(1, 2)$ by putting $x=\xi+1$, $y=\eta+2$, and then use Ascending Division; the linear approximation to η gives the tangent (§ 99).

10. Find the equations of the tangents to the following curves at the points specified and the shapes of the graphs near the points:

(i) $y=(2x^2-x+1)/(x^2-x+2)$ at $(1, 1)$;

(ii) $y=x(x-1)/(x-2)^2$ at $(3, 6)$;

(iii) $y=x^2(x-1)/(x+1)$ at $(2, \frac{4}{3})$;

(iv) $y=(1+x+x^3)/(1-x+x^2)$ at $(1, 3)$;

(v) $y=(10x^2-38x+37)/(3x^2-10x+9)$ at $(2, 1)$.

11. Use the methods of change of origin and successive approximation to draw the graphs of the following equations, plotting guiding points to give precision to the figure:

(i) $y=\frac{x-2}{x-1}$; (ii) $y=\frac{(x-2)(x-3)}{x-1}$; (iii) $y=\frac{(x-2)^2}{x-1}$;

(iv) $y=\frac{x-2}{(x-1)^2}$; (v) $y=\frac{x^2(x-2)}{(x-1)^2}$; (vi) $y=\frac{x^2(x-2)^2}{(x-1)^3}$.

12. Verify the following table for the equation $y=\frac{x^2+x+1}{x^2-x+1}$, and use it to draw the graph of the equation:

x	small	large	-3	-2	-1	1	2	3
y	$1+2x+2x^2$, app.	$1+2/x$, app.	7/13	3/7	1/3	3	7/3	13/7

13. Verify the following table for the equation $y=\dfrac{(x-1)(x-2)}{x^2-x+1}$, and use it to draw the graph of the equation :

x	nearly 1	nearly 2
	$\eta=-\xi+2\xi^2$, app.	$\eta=\xi/3-\xi^3/9$, app.

x	small	large	3/2	4	$-1/2$	-4
y	$2-x-2x^2$, app.	$1-2/x$, app.	$-1/7$	6/13	15/7	10/7

14. Verify the following table for the equation $y=\dfrac{x^2-x+1}{(x-1)(x-2)}$, and use it to draw the graph of the equation :

x	nearly 1	nearly 2
	$\eta=-1/\xi$, app.	$\eta=3/\xi$, app.

x	small	large	$-1/2$	-4	3/2	4
y	$1/2+x/4+5x^2/8$, app.	$1+2/x$, app.	7/15	7/10	-7	13/6

15. Verify the following table for the equation $y=\dfrac{(x-1)(x-2)}{(x-3)(x-4)}$, and use it to draw the graph of the equation :

x	nearly 1	nearly 2	nearly 3	nearly 4
	$\eta=-\xi/6+\xi^2/36$, app.	$\eta=\xi/2+5\xi^2/4$, app.	$\eta=-2/\xi$, app.	$\eta=6/\xi$, app.

x	large	0	-1	3/2	5/2	7/2	5	10
y	$1+4/x$, app.	1/6	3/10	$-1/15$	1	-15	6	12/7

16. Prove that (2, 1) is a point of inflexion on the graph of

$$y=-x^3+6x^2-11x+7,$$

and find the equation of the inflexional tangent.

17. Find the point of inflexion on the graph of

$$y=2x^3-3x^2-12x+18,$$

and show that the inflexional tangent meets the graph in three coincident points.

18. Prove that the line $y=2(x+1)$ meets the graph of

$$y=x^3-3x^2+5x+1$$

in three coincident points. Draw the line and the graph in the neighbourhood of the point of contact.

19. Prove that the point (3, 0) is a point of inflexion on the graph of $y=\dfrac{x^2-5x+6}{x^2-3x+3}$; find the equation of the inflexional tangent, and draw the graph and the tangent in the neighbourhood of the point of contact.

20. In the equation $y=\dfrac{8x^2-24x+22}{4x^2-8x+7}$, put $x=\xi+h$, $y=\eta+k$; use Ascending Continued Division, and determine h, k so that the point (h, k) may be a point of inflexion on the graph of

$$y=(8x^2-24x+22)/(4x^2-8x+7).$$

CHAPTER XIV.

DERIVATIVES OF POLYNOMIALS. MAXIMA AND MINIMA.

102. Gradient of a Graph. In tracing a graph accurately it is almost essential to know the position of the turning points. When a polynomial is expressed in factors it is fairly easy to find, at least approximately, the position of the turning points, but the case is altered when the polynomial is not in factor form.

The student will have noticed that at a turning point on the graph of a polynomial the tangent is parallel to the x-axis, though the tangent at a point may be parallel to the x-axis and yet the point not be a turning point. For example, the x-axis is a tangent to the graph of x^3 at the origin, but the origin is a point of inflexion, not a turning point. We must, however, seek for the turning points among those at which the tangent is parallel to the x-axis, and we therefore give now a method of finding an expression for the *gradient* of the tangent at any point on the graph of a polynomial. The gradient of the tangent at a point on a graph is often called the **gradient of the graph** at that point; we shall see how much additional power in forming a mental picture of the graph is to be obtained by a knowledge of the gradient.

When the origin is shifted to a point on the curve, the equation takes the form

$$\eta = a\xi + b\xi^2 + \text{higher powers of } \xi,$$

and the coefficient of ξ in this equation is the gradient of the tangent at the *new origin*. We shall apply this transformation to the graph of a polynomial.

I. $$y=ax^2+bx+c.$$

Shift the origin to the point (h, k) on the graph by writing $h+\xi$ for x and $k+\eta$ for y; we obtain

$$\begin{aligned} k+\eta &= a(h+\xi)^2+b(h+\xi)+c \\ &= ah^2+bh+c+(2ah+b)\xi+a\xi^2. \end{aligned}$$

But $k=ah^2+bh+c$ since (h, k) is on the curve, so that we have

$$\eta=(2ah+b)\xi+a\xi^2. \quad\dots\dots\dots\dots(1)$$

The gradient at the new origin is therefore $(2ah+b)$, that is, the gradient at the point (h, k) when the curve is referred to the old axes is $(2ah+b)$. But (h, k) is any point on the curve; we may therefore write x, y in place of h, k, and we now have the

Rule. *The gradient of the graph of $y=ax^2+bx+c$ at any point on it whose abscissa is x is*

$$2ax+b. \quad\dots\dots\dots\dots(\mathrm{I}a)$$

II. $$y=ax^3+bx^2+cx+d.$$

Shift the origin to any point (h, k) on the curve; we obtain

$$\begin{aligned} k+\eta &= a(h+\xi)^3+b(h+\xi)^2+c(h+\xi)+d \\ &= (ah^3+bh^2+ch+d)+(3ah^2+2bh+c)\xi+(3ah+b)\xi^2+a\xi^3, \end{aligned}$$

and therefore, since $k=ah^3+bh^2+ch+d$,

$$\eta=(3ah^2+2bh+c)\xi+(3ah+b)\xi^2+a\xi^3. \quad\dots\dots\dots(2)$$

Hence, the gradient at any point (h, k) on the curve is $(3ah^2+2bh+c)$, and we can state the

Rule. *The gradient of the graph of $y=ax^3+bx^2+cx+d$ at any point on it whose abscissa is x is*

$$3ax^2+2bx+c. \quad\dots\dots\dots\dots(\mathrm{II}a)$$

III. $$y=ax^n+bx^{n-1}+cx^{n-2}+\dots+px^2+qx+r,$$

where n is a positive integer.

The student who knows the Binomial Theorem will have no difficulty in proving, by the same method as in I and II

that the gradient of the graph of equation III at any point on it whose abscissa is x is

$$nax^{n-1}+(n-1)bx^{n-2}+(n-2)cx^{n-3}+\dots+2px+q. \quad \dots\text{(IIIa)}$$

Equations I, II are the cases of III for which $n=2$, $n=3$ respectively. The following points should be noted:

(i) The *absolute term*, c in I, d in II, r in III, does not appear in the gradient. (What is the graphical explanation of this?)

(ii) Any term in the gradient is obtained from the corresponding term in the polynomial by multiplying by the index of the power of x in that term and then subtracting 1 from that index.

Thus the term $2bx$ in (IIa) is obtained from the term bx^2 in II by multiplying bx^2 by 2 (which gives $2bx^2$) and then subtracting 1 from the index 2 (which gives $2bx$). The terms bx, cx, qx in I, II, III give b, c, q respectively.

Ex. Verify the expression for the gradient in the following cases:

Polynomial.	Gradient.
x.	1.
x^2.	$2x$.
x^3.	$3x^2$.
x^n.	nx^{n-1}.
$ax+b$.	a.
x^2-2x-1.	$2x-2$.
$3x^2-5x-7$.	$6x-5$.
$10+17x-4x^2$.	$17-8x$.
x^3-5x^2+9.	$3x^2-10x$.
$2x^4+16x^3-2x+1$.	$8x^3+48x^2-2$.
$(x-1)(x-2)(x-3)$.	$3x^2-12x+11$.
$x(x^2-1)(x^2-4)$.	$5x^4-15x^2+4$.
$(x-a)^2(x-b)$.	$(x-a)(3x-a-2b)$.
$(x-a)^n(x-b)$.	$(x-a)^{n-1}\{(n+1)x-a-nb\}$.

103. Derivatives. The expressions that have been found for the gradient are called **the derived functions** or, more briefly, **the derivatives** of the polynomials. Thus $2ax+b$ is the derivative of ax^2+bx+c.

When the polynomial is represented by a single letter, as y, or by a functional symbol, as $f(x)$ or $F(x)$, the

derivative is often denoted by the single letter with an accent, as y', or by the functional symbol with an accent on the functional letter, as $f'(x)$ or $F'(x)$.

Thus if $\quad y=ax^2+bx+c,\;$ then $\quad y'=2ax+b;$

if $\;f(x)=ax^2+bx+c,\;$ then $\;f'(x)=2ax+b.$

Sometimes the derivative is expressed by the letter D (the first letter of the word "derivative") placed to the left of the polynomial, which is enclosed in brackets; thus

$$D(ax^2+bx+c)=2ax+b.$$

The value when $x=a$ of the derivative $f'(x)$ is denoted by $f'(a)$.

Thus if $\qquad f(x)=4x^3-5x^2+2,$

then $\;f'(x)=12x^2-10x;\;\; f'(a)=12a^2-10a;\;\; f'(1)=2;\;\; f'(\tfrac{1}{2})=-2.$

If the independent variable of the function (or the abscissa of the curve) is denoted by some other letter than x, say by u or t, the derivative is of course formed by the same rule; for example,

if $\qquad y=au^2+bu+c, \quad y'=Dy=2au+b;$

and if $\quad y=at^2+bt+c, \quad y'=Dy=2at+b.$

When we wish to indicate in the symbols y', Dy which letter is used for the independent variable, that letter is placed as a suffix: thus, y'_x, D_xy, y'_u, D_uy.

Ex. 1. If $f(x)=x^3-3x+1$, for what values of x is $f'(x)$ zero?

$$f'(x)=3x^2-3=3(x-1)(x+1).$$

The values of x for which $f'(x)$ is zero are therefore the roots of the equation

$$3(x-1)(x+1)=0.$$

The required values are therefore 1 and -1.

Ex. 2. If $f(x)=x(x-1)(x-2)$, for what values of x is $f'(x)$ zero?

$$f(x)=x^3-3x^2+2x, \quad f'(x)=3x^2-6x+2.$$

The required values are the roots of the equation

$$3x^2-6x+2=0,$$

namely $\dfrac{3\pm\sqrt{3}}{3}$, that is, 1·577 and 0·423 approximately.

Ex. 3. Find the equation of the tangent at the point (1, 5) on the curve
$$y=x^3-5x^2+9. \qquad \text{(i)}$$
The equation of the line through (1, 5) with gradient m is
$$y-5=m(x-1). \qquad \text{(ii)}$$
The gradient y' at the point on the curve whose abscissa is x is given by the equation
$$y'=3x^2-10x,$$
and this equation gives $y'=-7$ when $x=1$. The value of m in (ii) is therefore -7, and the required equation is
$$y-5=-7(x-1) \quad \text{or} \quad 7x+y=12.$$
In the same way the equation of the tangent at any other point on the curve may be found. Thus at the point on the curve whose abscissa is 2, that is, at the point $(2, -3)$ on the curve, the value of y' is -8; the tangent at $(2, -3)$ is therefore the line through $(2, -3)$ with gradient -8, and its equation is
$$y+3=-8(x-2) \quad \text{or} \quad y=-8x+13.$$

Ex. 4. Find the equations of the tangents to the following curves at the points on the curves whose abscissae are given:

(i) $y=3x^3-5x-7$; $x=1$, $x=3$.

(ii) $2py=x^2$; $x=2p$, $x=h$.

(iii) $y=(x-1)(x-2)(x-3)$; $x=0, 1, 2, 3, 4, -1, -2$.

(iv) $y=(x-a)^2(x-b)$; $x=a$, $x=b$.

(v) $a^{n-1}y=x^n$; $x=0$; $x=b$, $y=c$.

If $a^{n-1}y=x^n$, then $y'=\dfrac{nx^{n-1}}{a^{n-1}}=\dfrac{nx^n}{a^{n-1}x}=\dfrac{ny}{x}$.

When $x=b$ and $y=c$ the gradient is nc/b, and the tangent is
$$y-c=\frac{nc}{b}(x-b) \quad \text{or} \quad by=ncx-(n-1)bc.$$
The value of c is, of course, b^n/a^{n-1}, but the equation is often more useful when it contains both the coordinates of the point of contact.

Ex. 5. The equation of the tangent at the point $(a, f(a))$ on the graph of $y=f(x)$ is
$$y-f(a)=(x-a)f'(a).$$
When $x=a$, $y=f(a)$; when $x=a$, the gradient $f'(x)$ is equal to $f'(a)$. The tangent is therefore the line through $(a, f(a))$ with gradient $f'(a)$.

104. Use of the Derivative in Curve Tracing. We shall work one or two examples to illustrate the use of the derivative in curve tracing. We first note the following properties of a curve that must be familiar to the student.

(1) If the gradient at a point P on a curve is positive, the tangent at P has a right-hand upward slope; a point

moving along the curve in the direction of increasing abscissa (from left to right) will as it passes through P be moving *upwards* as well as to the right. On the other hand, if the gradient at P is negative, the point will be moving *downwards* as well as to the right when it passes through P. Hence if the tracing point move so that its abscissa increases, it will move upwards or downwards according as the derivative is positive or negative.

(2) If the gradient at P is zero the tangent at P is parallel to the x-axis; the tracing point is for the moment moving neither up nor down, and P is, as a rule, a turning point.

If immediately to the left of P the gradient is positive and immediately to the right negative, then the point rises as it approaches P and descends after passing P; P is a turning point, and the ordinate at P is said to be a **maximum.** The ordinate at P is greater (algebraically) than any other ordinate near P and on *either* side of P.

On the other hand, if in approaching P the gradient is negative and after passing P positive, the tracing point first descends and then ascends; P is still a turning point, but the ordinate is now a **minimum**—less (algebraically) than any other ordinate near P and on *either* side of P. (See the remark in Ex. 1.)

The gradient at P, however, may be zero and yet P may not be a turning point; it may be a point of inflexion. As a rule inspection of the ordinate near P, or, preferably, of the gradient near P, enables us to decide easily whether P is a turning point or not.

Ex. 1. Find the turning points on the graph of

$$y=x(x-1)(x-2).$$

The graph is shown in Fig. 96, p. 242. In § 103, Ex. 2, we have seen that the gradient is zero when $x=1{\cdot}577$ and when $x=0{\cdot}423$; the corresponding values of y are $-0{\cdot}385$ and $0{\cdot}385$. The points $S'(1{\cdot}577, -0{\cdot}385)$ and $R'(0{\cdot}423, 0{\cdot}385)$ are the turning points; at S' the ordinate is a minimum and at R' a maximum. These points R' and S' are near the points R and S of the figure.

Note that the ordinate at R' is not the *greatest* ordinate of the curve; it is a maximum because it is greater than *any other near it.* Similarly the ordinate at S' is a minimum, though it is obviously not the *least* ordinate of the curve.

Ex. 2. $$y = f(x) = 2x^3 - 3x^2 - 12x + 10.$$

Here $$f'(x) = 6x^2 - 6x - 12 = 6(x+1)(x-2),$$

and the gradient is zero when $x = -1$ and when $x = 2$. The points at which x has these values are probably turning points; in any case they are useful as *guide points* for the discussion of the graph.

Draw up the table:

x	$-\infty$	-3	$-2{\cdot}5$	-2	-1	0	1	2	3	4	$+\infty$
$f'(x)$	$+\infty$	60	$40{\cdot}5$	24	0	-12	-12	0	24	60	$+\infty$
y	$-\infty$	-35	-10	6	17	10	-3	-10	1	42	$+\infty$

It is quite clear from the table that the points $(-1, 17)$, $(2, -10)$ are turning points, but the table gives much more information. The table suggests that as x increases from $-\infty$ to -1 *the gradient is positive*, and therefore that the tracing point steadily rises from the extreme low left to the position $(-1, 17)$. Examination of the gradient confirms the suggestion; because if x is negative and numerically greater than 1 both $x+1$ and $x-2$ are negative, and therefore $f'(x)$, which is equal to $6(x+1)(x-2)$, is positive.

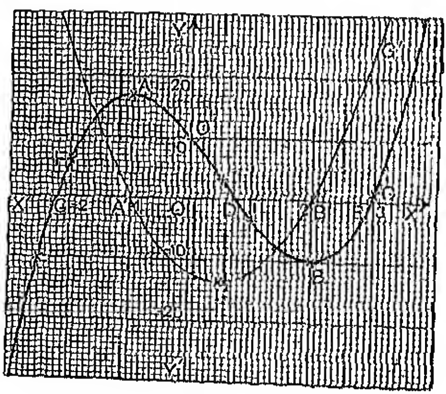


FIG. 98.

From $x = -1$ to $x = 2$ the gradient is *negative*; therefore the tracing point now descends till it reaches the position $(2, -10)$. *After passing this position the gradient is always positive*; therefore the tracing point now steadily rises and moves off to the right and upwards to infinity. We thus have a complete description of the way in which the point traces out the curve.

It is easy now to draw the curve. When plotting the points it is well to draw a short length of the tangent at the point; near its point of contact the tangent practically coincides with the curve. The graph is $CABG$ (Fig. 98).

We can also solve, approximately, the equation

$$2x^3-3x^2-12x+10=0;$$

the curve crosses the x-axis where x has the values $-2{\cdot}22$, $0{\cdot}76$ and $2{\cdot}96$ approximately, and these are the roots.

At a point of inflexion *the gradient has a turning value.* Thus, on the graph of x^3 (Fig. 68, p. 184), the gradient is positive, but decreases as we pass along the curve from P' to O, where it is zero; as we continue along the curve from O towards P the gradient is again positive and increases. It has therefore a turning value, zero, at the point of inflexion O. Similarly, in Fig. 90, p. 234, as we pass along the curve from P' to P the curve gets steeper as we approach O, but on passing O the curve becomes less steep; the gradient has again a turning value at O, not zero in this case.

Now when we have found the derivative of a function we can draw its graph, and to every turning point on this graph will correspond a point of inflexion on the graph of the given function. The graph of the derivative of a function is called the **derived** curve of the function; the graph of the function itself may be called the primitive curve.

Ex. 3. Find the point of inflexion on the graph of the equation

$$y=f(x)=2x^3-3x^2-12x+10.$$

The derivative of $f(x)$ is $6x^2-6x-12$ and the graph of this derivative, $F'I'G'$ in Fig. 98, is the derived curve of FIG. The points F and F' have the same abscissa, and these are called "corresponding points" of the two curves. Similarly A and A', I and I', etc., are corresponding points. The ordinate of F' measures the gradient at F; thus the gradient at F, where $x=-2$, is 24, and this is the ordinate of F'. (In plotting the derived curve it will usually be necessary to choose a new scale unit for the ordinates, but the scale unit for the abscissae should always be the same; both scales are the same in Fig. 98.)

Let us, for the moment, denote the gradient or derivative by g;

then $$g=6x^2-6x-12, \quad g'=12x-6.$$

Here $g'=0$ when $x=\frac{1}{2}$. The point $I'(\frac{1}{2}, -13\frac{1}{2})$ on the derived curve is a turning point on that curve; therefore the corresponding point $I(\frac{1}{2}, 3\frac{1}{2})$ on the primitive curve is a point of inflexion on that curve.

Similarly, the abscissa of the point of inflexion on the graph of Example 1 is given by $D(3x^2-6x+2)=0$ or $6x-6=0$ or $x=1$; the point $(1, 0)$ is therefore the point of inflexion.

The gradient, denoted above by g, is the derivative of the function y or $f(x)$, and therefore g', the derivative of g, is *the derivative of the derivative of* y; g' is called the *second derivative of* y. The second derivative of a function is denoted by two accents, y'' or $f''(x)$; thus, if

$$y=f(x)=2x^3-3x^2-12x+10,$$

then $$y'=f'(x)=6x^2-6x-12;\quad y''=f''(x)=12x-6.$$

We might in the same way form third and higher derivatives; thus in the above examples the third derivative is denoted by y''' or $f'''(x)$ and $y'''=12$.

In distinction from higher derivatives y' is called the *first* derivative.

Using the second derivative, we have now the following rule:

The abscissae of the points of inflexion on the graph of $f(x)$ are, in general, the roots of the equation $f''(x)=0$.

We say "in general," because a value of x may be a root of $f''(x)=0$ and yet not give a point of inflexion; for example, if $f(x)=x^4$, then $f''(x)=12x^2$, but the origin is not a point of inflexion on the graph of x^4. In all cases, however, the abscissa of a turning point on the derived curve is the abscissa of a point of inflexion on the primitive curve.

Note. The results given in questions 11 and 12 of *Exercises XXVII.* are very important. Thus, for No. 11, we have

$$k+\eta=\frac{a}{h+\xi}=\frac{a}{h}-\frac{a}{h^2}\xi+\frac{a\xi^2}{h^2(h+\xi)}.$$

But $k=a/h$ since (h, k) is on the graph of $y=a/x$, and therefore the first approximation near the new origin (§ 99) is

$$\eta=-\frac{a}{h^2}\xi,$$

so that the gradient there is $-a/h^2$. Hence the gradient at any point on the graph of a/x is $-a/x^2$. The function $-a/x^2$ is called the derivative of a/x. (Compare §§ 102, 108.)

The result in the case of $y=a/x^n$ shows that the derivative of $\frac{a}{x^n}$ is $-\frac{na}{x^{n+1}}$. If we use negative indices, then we have

$$D(ax^{-n})=-nax^{-n-1}$$

which shows that the rule for forming the derivative of a power (§ 102) holds for negative as well as for positive integral indices.

EXERCISES XXVII.

Trace the curves given by equations 1–10; state their turning points, their points of inflexion, and the values of x at which they cross or touch the x-axis.

1. $y=7x^2-12x-10.$
2. $y=15+6x-2x^2.$
3. $y=x^2(x-2).$
4. $y=2x^3+3x^2-12x+6.$
5. $y=x^3-3x+1.$
6. $y=x^3-7x+3.$
7. $y=x^4-8x^3+15x^2+4x-20.$
8. $y=3x^4-8x^3+6x^2-10.$
9. $y=x^4-4x^3-4x^2+16x+21.$
10. $y=x^5-5x^4+5x^3+10.$

11. Show by shifting the origin to the point (h, k) on the graph of $y=\frac{a}{x}$, that the equation takes the form

$$\eta=-\frac{a}{h^2}\xi+\text{higher powers of }\xi,$$

and then prove that the gradient of the graph of $y=\frac{a}{x}$ at any point on it whose abscissa is x is $-\frac{a}{x^2}$.

12. Show, as in Ex. 11, that the gradients of the graphs of

$$y=\frac{a}{x^2},\quad y=\frac{a}{x^3},\quad y=\frac{a}{x^n},$$

where n is a positive integer, are respectively

$$-\frac{2a}{x^3},\quad -\frac{3a}{x^4},\quad -\frac{na}{x^{n+1}}.$$

Deduce that the gradient of the graph of

$$y=a+\frac{b}{x}+\frac{c}{x^2}+\frac{d}{x^3}\quad\text{is}\quad -\frac{b}{x^2}-\frac{2c}{x^3}-\frac{3d}{x^4}.$$

13. Find the turning points on the graphs of equations 9–19, Exercises XXIII.

105. The Tangent and Coincident Points. In preceding sections we have found the equation of a tangent by using the method of successive approximations; there is another method that is of great use which we shall now consider.

Suppose a straight line L to meet a curve at two points A and B. Move L so that these points of intersection

come nearer and nearer to each other until they coincide, say at P; the line L is now a tangent and P is its point of contact. It does not matter of course how the points A and B are taken on the curve to begin with, provided they come together at P; both of them might be distinct from P (Fig. 99 (a)), or one of them, A say, might coincide with P (Fig. 99 (b)). The line L might also meet the curve at other points than A and B; for example, at C. The tangent will also in that case meet the curve at the point D, to which C has shifted.

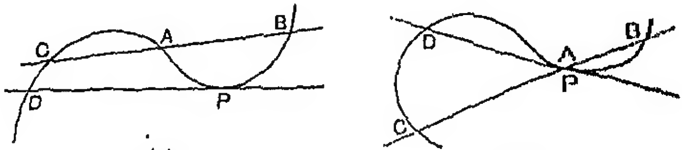


FIG. 99.

Let us now consider the algebraic counterpart of this way of regarding the tangent. The coordinates of the points A and B are obtained by solving the equations of the line and curve as simultaneous equations. Suppose for simplicity that the line and curve never intersect in more than two points; then so long as A and B are distinct the equations will give two unequal values for the x of the points, but when the points coincide at P there will be only one distinct value for x. *The equation for x will still be of the second degree, but it will have its two roots equal.*

Take, for example, the equations

$$y=5x+c, \ldots\ldots (1) \qquad y=2x^2-3x+1. \ldots\ldots (2)$$

Solving these as simultaneous equations, we find the equation for x,

$$2x^2-8x+1-c=0. \quad \ldots\ldots (3)$$

This equation gives the abscissae of the points in which the line (1) cuts the curve (2), and it has in general two unequal roots; to each root the equation (1) gives the corresponding value of y, so that we obtain the coordinates of the two points of intersection, (x_1, y_1), (x_2, y_2), say.

By giving different values to c, we make the line (1) move into different positions (the different lines being parallel in this case).

Now let us move the line until it becomes a tangent; the two roots

of equation (3) must then become equal, and we know that the roots of the equation will be equal if

$$64=8(1-c) \quad \text{or} \quad c=-7.$$

Equation (1) now becomes $y=5x-7$, and this is the equation of the tangent.

Note that when $c=-7$, equation (3) is still a quadratic equation,

$$2x^2-8x+8=0 \quad \text{or} \quad 2(x-2)(x-2)=0;$$

each root is now 2, and when $c=-7$ equation (1) gives for each of these equal values of x the equal values 3 and 3 for y. In other words, we now have $x_1=x_2=2$ and $y_1=y_2=3$, and (2, 3) is the point of contact.

Again, the points in which the line $y=3-4x$ intersects the curve

$$y=\frac{2x^2-x+3}{x^2+x+1}$$

are obtained by solving these as simultaneous equations. The equation for x is

$$(3-4x)(x^2+x+1)=2x^2-x+3 \quad \text{or} \quad x^2(4x+3)=0,$$

so that $x=0$ *twice* and $x=-\frac{3}{4}$ once. The solutions of the simultaneous equations are therefore $x=0$, $y=3$ *twice* and $x=-\frac{3}{4}$, $y=6$ once. The line therefore touches the curve at the point (0, 3) and intersects it again at the point $(-\frac{3}{4}, 6)$.

The conception of equal roots of an equation and of coincident points on a curve, though at first sight artificial, is really very natural. In general, a line meets a curve in two or more distinct points, and the equation that determines the x (or, if we please, the y) of the points has two or more distinct roots; but we may move the line so that two of the points become coincident, and then two roots of the equation become equal. The graphical interpretation of the coincidence of the points and the equality of the roots is that the line is now a tangent, though it may of course intersect the curve elsewhere. We are thus led to the following definition.

Definition. The tangent to a curve at a point P on it is a line which meets the curve in two coincident points at P.

The algebraical form of this definition is as follows:

If the equations of a straight line and a curve, regarded as simultaneous equations, have a solution which appears twice, then the straight line is a tangent, and the repeated solution gives the coordinates of the point of contact.

Ex. 1. Find the equation of the tangent at the point (1, 1) on the graph of the equation $y=2x^2-3x+2$. ....(i)

The equation of any straight line through (1, 1) is of the form

$$y-1=m(x-1). \qquad \text{(ii)}$$

Substituting from (ii) in (i), we have the equation for x,

$$2x^2-(m+3)x+(m+1)=0. \qquad \text{(iii)}$$

If equation (iii) has equal roots, we must have

$$(m+3)^2-8(m+1)=0 \quad \text{or} \quad (m-1)^2=0.$$

Therefore $m=1$ (twice). Putting 1 for m in (iii), we see that $x=1$ twice, and therefore by (ii) $y=1$ twice. The required equation is thus $y=x$.

Why should the equation for m give $m=1$ twice? The reason is that in general we can draw *two* tangents to the graph of (i) from a given point, but if, as in this case, the given point is on the curve, the two tangents *coincide*. Compare Ex. 2.

Ex. 2. Find the equations of the tangents from the point (2, 2) to the graph of the equation

$$y=2x^2-3x+2. \qquad \text{(i)}$$

The point (2, 2) is not on the curve. Any line through (2, 2) is given by $y-2=m(x-2)$. ....(ii)

Solving (i) and (ii) as simultaneous equations, we get for the abscissae of the points in which line and curve intersect,

$$2x^2-(m+3)x+2m=0. \qquad \text{(iii)}$$

The line will be a tangent if the roots of (iii) are equal, and the condition for equal roots is

$$(m+3)^2-16m=0 \quad \text{or} \quad (m-1)(m-9)=0,$$

so that $m=1$ or 9.

If $m=1$, equation (iii) gives $x=1$ twice and then (ii) gives $y=1$ twice; one tangent is therefore $y=x$, the point of contact being (1, 1).

If $m=9$, equation (iii) gives $x=3$ twice, and then (ii) gives $y=11$ twice; the other tangent is therefore $y=9x-16$, the point of contact being (3, 11).

Ex. 3. Find the equation of the tangent at (1, 0) to the graph of the equation $y=x(x-1)(x-2)$. ....(i)

Any line through (1, 0) is given by

$$y=m(x-1). \qquad \text{(ii)}$$

Solving (i) and (ii) for the abscissae of the points of intersection we get the equation

$$m(x-1)=x(x-1)(x-2) \quad \text{or} \quad (x-1)(x^2-2x-m)=0. \qquad \text{(iii)}$$

One root of (iii) is 1, and as the line is to be a tangent at the poin

a second root of (iii) must be 1. But 1 will be a root of $-m=0$ if $m=-1$. Equation (iii) now becomes

$$(x-1)(x-1)(x-1)=0,$$

in this case there are *three* equal roots. The point (1, 0) is of inflexion and $y=-x+1$ is the inflexional tangent.

we see that the inflexional tangent meets the curve in *three* ent points at the point of inflexion.

.. Find the equation of the secant through the points (x_1, y_1) $, y_2)$ on the conic $$ax^2+by^2=1, \text{ ....(i)}$$

luce the equation of the tangent at (x_1, y_1).

equation of the line through (x_1, y_1) and (x_2, y_2) is

$$\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2}. \text{ ....(ii)}$$

introduce the condition that the points lie on the conic (i);

o $$ax_1^2+by_1^2=1, \quad ax_2^2+by_2^2=1,$$

erefore $$a(x_1^2-x_2^2)=-b(y_1^2-y_2^2). \text{ ....(iii)}$$

iplying the left side of equation (ii) by $a(x_1^2-x_2^2)$ and the ide by the equal quantity $-b(y_1^2-y_2^2)$, we get

$$a(x_1+x_2)(x-x_1)=-b(y_1+y_2)(y-y_1),$$

$$a(x_1+x_2)x+b(y_1+y_2)y=ax_1^2+by_1^2+ax_1x_2+by_1y_2$$

$$a(x_1+x_2)x+b(y_1+y_2)y=1+ax_1x_2+by_1y_2, \text{ .... (iv)}$$

$ax_1^2+by_1^2=1$. (The student should verify equation (iv) by g that this linear equation is true provided (x_1, y_1) and (x_2, y_2) he conic; the equation is therefore independent of the partiethod by which it was obtained.)

nd the tangent put $x_2=x_1$ and $y_2=y_1$ in equation (iv); we get

$$2ax_1x+2by_1y=1+ax_1^2+by_1^2=1+1$$

$$ax_1x+by_1y=1.$$

5. Determine c so that the straight line

$$3x-2y+c=0 \text{ ....(i)}$$

e a tangent to the curve given by the freedom equations

$$x=t(t-1), \quad y=t(t+1). \text{ ....(ii)}$$

values of t for the points of intersection of line and curve are ed by substituting in (i) the values of x and y given by (ii); thus

$$3t(t-1)-2t(t+1)+c=0 \quad \text{or} \quad t^2-5t+c=0. \text{ ....(iii)}$$

roots of (iii) will be equal if the line is a tangent. Hence , and the equation of the tangent is

$$12x-8y+25=0.$$

point of contact is given by equations (ii) when $t=5/2$, the root of (iii) when $c=25/4$; the point is $(\frac{15}{4}, \frac{35}{4})$.

106. Some Theorems on Roots of Equations. The following theorems are often needed in discussing tangents and turning values.

The Quadratic Equation. It is proved in all books on elementary algebra that the quadratic equation

$$ax^2+bx+c=0$$

has its roots (i) real and unequal, (ii) real and equal, (iii) imaginary, according as (b^2-4ac) is (i) positive (not zero), (ii) zero, (iii) negative. This expression b^2-4ac is called the **discriminant** of the quadratic equation.

The Cubic Equation. If we have the cubic equation in the form

$$x^3+qx+r=0, \quad \text{.........................}(c)$$

the expression $$4q^3+27r^2$$

is called the **discriminant** of the equation. A cubic equation has always at least one real root (the coefficients q, r being supposed real). Two of the roots will be equal when the discriminant is zero and the value of each of the equal roots is $-3r/2q$.

We may prove this theorem as follows. If equation (c) has two equal roots, then x^3+qx+r must have a squared factor, $(x-k)^2$, say. Divide x^3+qx+r by $(x-k)$; the integral quotient is $x^2+kx+(k^2+q)$, and the remainder is k^3+qk+r, which must be zero. Therefore

$$k^3+qk+r=0. \quad \text{.................................}(i)$$

The quotient must be again exactly divisible by $(x-k)$, and therefore the new remainder, $3k^2+q$, must be zero, so that we have

$$3k^2+q=0 \quad \text{or} \quad k^2=-\tfrac{1}{3}q. \quad \text{.........................}(ii)$$

But, by (i), $k(k^2+q)=-r$, and therefore, by (ii), $k=-3r/2q$. Substitution in (ii) now gives

$$\frac{9r^2}{4q^2}=-\tfrac{1}{3}q \quad \text{or} \quad 4q^3+27r^2=0.$$

It is easy to show that when $4q^3+27r^2=0$ we have

$$x^3+qx+r=\left(x+\frac{3r}{2q}\right)^2\left(x-\frac{3r}{q}\right),$$

so that the roots of equation (c) are $-\dfrac{3r}{2q}$, $-\dfrac{3r}{2q}$, $\dfrac{3r}{q}$.

When the discriminant is *negative* the three roots of the cubic (c) are real and different, but when the discriminant is *positive* two roots are imaginary and one real.

These results may be proved in the following way.

If y is a turning value of x^3+qx+r, then (§ 107) the equation

$$x^3+qx+r-y=0, \quad \text{.............................(1)}$$

considered as an equation in x, must have two equal roots. The point (x, y) is thus a turning point on the graph of

$$y=x^3+qx+r;$$

therefore we must have $Dy=0$, that is, $3x^2+q=0$, ...(2)

so that, when (x, y) is a turning point, equations (1) and (2) hold simultaneously, and the turning values are the values of y given by equations (1) and (2).

To find y we have, first by (1) and then by (2),

$$(r-y)^2=x^6+2qx^4+q^2x^2=-\frac{q^3}{27}+\frac{2q^3}{9}-\frac{q^3}{3},$$

or

$$y^2-2ry+\frac{1}{27}(4q^3+27r^2)=0. \quad \text{.........................(3)}$$

If y_1 and y_2 are the two roots of (3), it will be readily seen from graphical considerations that equation (c) will have three real and distinct roots if, and only if, y_1 and y_2 have opposite signs. Hence in this case the product y_1y_2 must be negative, or, since the factor 1/27 is positive, $4q^3+27r^2$ must be negative.

If equation (c) has two imaginary roots, y_1 and y_2 must be of the same sign, so that the product y_1y_2 and therefore also $4q^3+27r^2$ must be positive.

If equation (c) has two equal roots, either y_1 or y_2 must be zero, so that the product y_1y_2 and therefore also $4q^3+27r^2$ must be zero, as has been proved otherwise.

The cubic equation

$$x^3+px^2+qx+r=0 \quad \text{.....................(c')}$$

becomes, when $\xi-\frac{1}{3}p$ is put for x,

$$\xi^3+(q-\tfrac{1}{3}p^2)\xi+(\tfrac{2}{27}p^3-\tfrac{1}{3}pq+r)=0.$$

The discriminant of this cubic, and therefore also of the cubic (c′) is $\quad 4(q-\frac{1}{3}p^2)^3+27(\frac{2}{27}p^3-\frac{1}{3}pq+r)^2,$

which is equal to

$$4q^3+27r^2+4p^3r-p^2q^2-18pqr.$$

If this expression is negative, the roots of (c′) are all real and different; if it is zero two of the roots are equal, and if it is positive two are imaginary and one real.

The graph of ax^2+bx+c will intersect the x-axis in two

different points if the roots of the equation $ax^2+bx+c=0$ are real and different; it will touch the axis if the roots are equal, and will not intersect the axis at all if the roots are imaginary. But just as we say that the equation has two imaginary roots instead of saying that it has no roots, so it is convenient to say that the curve in this case cuts the x-axis in two "imaginary points," the abscissae of these points being the imaginary roots of the equation $ax^2+bx+c=0$. In a similar way curves are said to intersect in "imaginary points" when the equations that determine the coordinates of their points of intersection have imaginary roots. For example, the circle and the straight line given by the equations

$$x^2+y^2=6, \quad x+y=4$$

intersect in the two imaginary points $(2+\sqrt{-1},\ 2-\sqrt{-1})$ and $(2-\sqrt{-1},\ 2+\sqrt{-1})$.

If an equation with real coefficients is satisfied by the imaginary number $a+b\surd(-1)$, it is also satisfied by the *conjugate* imaginary $a-b\surd(-1)$. Hence, if the imaginary point $(a+b\sqrt{-1},\ c+d\sqrt{-1})$ lies on a real curve, so does the *conjugate* imaginary point $(a-b\sqrt{-1},\ c-d\sqrt{-1})$. It is easy to show that the line joining the two conjugate points is real, the equation of the line being formed by the same rule as when the points are real; the equation is

$$d(x-a)=b(y-c).$$

107. Turning Values. Maxima and Minima. We may discriminate maximum and minimum turning values of an ordinate by the following considerations. When a straight line is drawn parallel to the x-axis it will usually cut a curve in two or more points. Now suppose such a line to move up or down while remaining always parallel to the x-axis. When the line approaches a turning point of the curve two of the points of intersection come near each other, and when the line reaches the turning point these two points of intersection will become coincident; the line will *ascend* to reach a maximum and *descend* to reach a minimum turning point. The ordinate of the line when the two points of intersection coincide has a turning

value; if *further ascent* causes the two points of intersection to become imaginary, the turning value is a maximum, while if *further descent* causes the points of intersection to become imaginary the turning value is a minimum.

Translating these graphical considerations into analytical form, we have the following rule:

Let $f(x)$ be a given function of x. Find a value of y such that the equation $f(x)-y=0$, regarded as an equation in x, may have equal roots; if y_1 be any such value, and if, on $\begin{matrix}\text{increasing}\\\text{decreasing}\end{matrix}$ y_1 a little, two (or an even number) of the roots of the corresponding equation $f(x)-y=0$ become imaginary, then y_1 is a $\begin{matrix}\text{maximum}\\\text{minimum}\end{matrix}$ turning value of $f(x)$.

As has been pointed out in § 104, the tangent at a turning point is parallel to the x-axis, but it is possible for the tangent at a point to be parallel to the x-axis, and yet the point may not be a turning point.

108. Calculation of Turning Values. The following examples show how the above rules are applied. It may be noted that the turning value of a quadratic function ax^2+bx+c is the ordinate of the vertex of the parabola which is the graph of the function (§ 95, Ex. 1).

Ex. 1. Find the turning values of $x(x-1)(x-2)$.

Let $y=x(x-1)(x-2)=x^3-3x^2+2x$; then the equation to be considered is

$$x^3-3x^2+2x-y=0.$$

Comparing this equation with equation (c′) of § 106, we see that $p=-3$, $q=2$ and $r=-y$, so that the discriminant, D say, is

$$D=32+27y^2+108y-36-108y=27y^2-4.$$

$D=0$ if $y=\pm 2/3\sqrt{3}$, and D becomes positive when y becomes greater than $2/3\sqrt{3}$, and also when y becomes less (algebraically) than $-2/3\sqrt{3}$.

The turning values are therefore $2/3\sqrt{3}$ (a maximum) and $-2/3\sqrt{3}$ (a minimum). Compare § 104, Example 1.

Ex. 2. Find the turning values of $\dfrac{2x^2-x+3}{x^2+x+1}$.

Let $y=\dfrac{2x^2-x+3}{x^2+x+1}$, and treat this as an equation in x, namely,

$$(2-y)x^2-(1+y)x+(3-y)=0.$$

The discriminant D of this equation is

$$D=(1+y)^2-4(2-y)(3-y)=-3y^2+22y-23.$$

The roots of the equation $D=0$ are 6·07 and 1·26 approximately, so that

$$D=-3(y-6{\cdot}07)(y-1{\cdot}26).$$

The graph of $-3(y-6{\cdot}07)(y-1{\cdot}26)$ is an inverted festoon, the *abscissa* of any point on this graph being denoted by y. D becomes negative when y becomes greater than 6·07, so that 6·07 is a maximum turning value; D becomes negative when y becomes less than 1·26, so that 1·26 is a minimum turning value.

In discussing the *sign* of a quadratic function the method explained in the Examples to § 95 will be found useful.

Ex. 3. An open tank is to be constructed with a square base and vertical sides to hold a given quantity, a cub. ft., of water; show that the expense of lining the tank with lead will be least when the depth is half the width.

Let the side of the square base be x ft. and the depth of the tank y ft.; the capacity of the tank will be x^2y cub. ft., and this is constant and equal to a cub. ft., so that x and y are connected by the equation

$$x^2y=a. \qquad \text{.........(i)}$$

The expense of lining the tank is directly proportional to the surface to be covered, and this surface is (x^2+4xy) sq. ft.; we have therefore to find when (x^2+4xy) is a minimum. Denote this quantity by z, and substitute for y the value a/x^2 given by equation (i); we then have to consider the equation

$$z=x^2+\frac{4a}{x} \quad \text{or} \quad x^3-zx+4a=0. \qquad \text{.........(ii)}$$

Comparing with equation (c) of § 106, we find $q=-z$, $r=4a$, so that the discriminant D is given by the equation

$$D=-4z^3+432a^2=4(108a^2-z^3). \qquad \text{.........(iii)}$$

$D=0$ when $z=\sqrt[3]{(108a^2)}$ and, when z is a little less than $\sqrt[3]{(108a^2)}$, D is positive, so that $\sqrt[3]{(108a^2)}$ is the minimum value of z.

Denote the minimum value by z_1. When $z=z_1$ equation (ii) has two equal roots, and the value of each of these is $\dfrac{-12a}{-2z_1}$ or $\dfrac{6a}{z_1}$ (§ 106).

Let $x_1=6a/z_1$, then the corresponding value y_1 of y is a/x_1^2; y_1 and x_1 give the depth and the width when the expense of lining is least. Now

$$\frac{y_1}{x_1}=\frac{x_1^2y_1}{x_1^3}=\frac{a}{x_1^3}=\frac{az_1^3}{216a^3}=\frac{108a^3}{216a^3}=\frac{1}{2}$$

by inserting the values of x_1 and z_1. Thus $y_1=\frac{1}{2}x_1$.

In the next set of Exercises various examples are given which require the formation of an algebraic expression like that denoted by z in Example 3 above; indeed the chief

difficulty of such problems usually lies in the correct choice of the independent variable x. When the gradient can be found, the procedure shown in § 104 for finding turning values may be used.

Thus, in the tank problem, we have, by equation (ii),

$$z = x^2 + \frac{4a}{x},$$

and we find for the derivative of z,

$$Dz = 2x - \frac{4a}{x^2} = \frac{2(x^3 - 2a)}{x^2}.$$

$Dz = 0$ when $x = \sqrt[3]{(2a)}$, and Dz changes from negative to positive as x changes from a value that is a little less than $\sqrt[3]{(2a)}$ to a value that is a little greater than $\sqrt[3]{(2a)}$. Hence z is a minimum when $x = \sqrt[3]{(2a)}$. But, by (i), when $x = \sqrt[3]{(2a)}$ we find $y = \frac{1}{2}\sqrt[3]{(2a)}$. Therefore when z is a minimum, $y = \frac{1}{2}x$, or the depth is half the width.

EXERCISES XXVIII.

1. Calculate the turning value of $x^2 - 2x - 1$ (see Fig. 70).

2. Prove that the graph of $y = (2 - x)/(x - 1)$ has no turning value (see Fig. 77).

3. Calculate the turning values of the following functions, and the corresponding values of x:

(i) $x^2 - x$ (Fig. 89); (ii) $x + 1 + \dfrac{1}{x}$ (Fig. 94);

(iii) $\dfrac{x(x-1)}{x-2}$; (iv) $\dfrac{x^2 + x + 1}{x^2 - x + 1}$;

(v) $\dfrac{(x-1)(x-2)}{x^2 - x + 1}$; (vi) $\dfrac{x^2 - x + 1}{(x-1)(x-2)}$;

(vii) $\dfrac{(x-1)(x-2)}{(x-3)(x-4)}$.

4. Calculate the turning values of the following functions:

(i) $x^2(x-2)$ (Fig. 73); (ii) $x + 1 - \dfrac{2}{x^2}$ (Fig. 94);

(iii) $\dfrac{x^3 - 3x + 2}{x^2 - x + 1}$ (Fig. 95).

5. If (h, k) is a turning point on the graph of $y = f(x)$, find the forms of the first approximations to the equation of the graph when the origin is shifted to the point (h, k), (1) when k is a maximum value, (2) when k is a minimum value.

6. Shift the origin of the graph of $y=x^2(x-3)$ to the point $(2, -4)$, and then calculate the turning values of $x^2(x-3)$, stating which is a maximum and which is a minimum.

7. What change of origin will transform the equation

$$y=(x-1)(x-2)(x-3)$$

into an equation of the form $\eta=\xi^3+q\xi$? Find the point of inflexion on the graph of the equation, and calculate the maximum and minimum values of $(x-1)(x-2)(x-3)$.

8. If $(x-k)^2$ is a squared factor of $x^3-4x^2-y=0$, calculate the value of k, and the corresponding values of y. Hence find the turning values of the graph of $y=x^2(x-4)$; and determine from a rough graph which is a maximum and which is a minimum.

9. Find the discriminant of the equation $y=\dfrac{x^2+x+1}{x^2-x+1}$ regarded as an equation in x, where y is known. Draw a rough graph of how the discriminant varies as y varies, and find the turning values of $(x^2+x+1)/(x^2-x+1)$.

10. Find the discriminant of the equation

$$y=(x^2+38x+1)/(x^2+6x+1)$$

regarded as an equation in x, where y is known. Draw a rough graph of how the discriminant varies as y varies, and find the maximum and minimum values of $(x^2+38x+1)/(x^2+6x+1)$.

11. Find the greatest rectangle that can be inscribed in a triangle ABC of base a and height h, one side of the rectangle lying along BC and two vertices falling on AB, AC respectively.

12. A shepherd has a length l of netting with which to fence three sides of a rectangular piece of a field, the fourth side being formed by a dyke already made. Find the dimensions of the rectangle which contains the greatest area.

If part of the netting has to be used to divide the area into two rectangles, the division being at right angles to the dyke, what would be the dimensions for the greatest area?

13. A length l of wire is to be cut into parts; one part is to be bent into the form of a circle and the other into the form of a square. In what ratio must the wire be cut if the sum of the areas of the circle and the square is the least possible?

14. From two points A, B on a straight line two straight lines AX, BY are drawn perpendicular to AB and on the same side of AB; C is a point between A and B such that $AC=a$ and $CB=b$; from C two straight lines CD, CE are drawn at right angles to each other to meet AX at D and BY at E. If $AD=x$, find the value of x

when $AD+BE$ is a minimum;

when DE is a minimum;

when $AD+DE+BE$ is a minimum;

when the area of the trapezium $ADEB$ is a minimum;

when the sum of the areas of the triangles ADC, BCE is a minimum;

when the area of the triangle DCE is a minimum.

ough the point $A(a, b)$ in the first quadrant a straight line utting the axes OX, OY on the positive side of the origin O spectively. Find

the minimum value of the area of the triangle OBC;

the minimum value of $OB+OC$;

the minimum value of $BA \cdot AC$;

the maximum value of $\dfrac{BA \cdot AC}{BC^2}$.

raight line of given length is divided into two parts so that square on one part with thrice the square on the other part possible; find the ratio of the two parts.

B, C, D are the vertices in order of a variable quadrilateral $AB=CD=a$, a constant, and $AC=BD=b$, a constant; $BC+AD$ is least when $ABCD$ is a rectangle.

perimeter of a triangle is given, and the length of one side at of another. Show that the ratio of the shortest side to ter lies between $\frac{1}{6}$ and $\frac{1}{4}$, and that the area is greatest when s $(11-\sqrt{13})/36$.

ve that $5x^3>4x^2-1$ for all positive values of x.

uss the inequality $\dfrac{3x-1}{x-1} \gtreqless 1$.

d the maximum and minimum values of the ordinate of

$$xy(x+y)+x^2+y^2=0.$$

ify that y is a maximum or minimum at the points $(1, 1)$, e curve

$$x^3+y^3-9xy+6x+7y-6=0,$$

minate between the alternatives.

estigate the maximum and minimum values of the following

$\dfrac{x^2-4x+7}{x^2-2x+4}$; (ii) $\dfrac{x^2+6x-11}{x^2+4x-5}$; (iii) $\dfrac{x^2+2x+3}{x^2+3x+2}$;

$\dfrac{3x^2+7x+2}{5x^2+8x+3}$; (v) $\dfrac{8x^2+10x+2}{x^2+2x+2}$; (vi) $\dfrac{x^2-2x+3}{x^2+2x+3}$,

24. If the expression

$$\frac{ax^2+2hx+b}{a'x^2+2h'x+b'}$$

be capable of all real values for real values of x, prove that

$$a'b'<h'^2 \quad \text{and} \quad (ab'-a'b)^2<4(a'h-ah')(bh'-b'h).$$

25. Prove that for real values of x, $\dfrac{ax^2+bx+c}{cx^2+bx+a}$ will be capable of all values whatever if $b^2>(a+c)^2$, that there will be two values between which it cannot lie if $4ac<b^2<(a+c)^2$, and two values between which it must lie if $b^2<4ac$.

CHAPTER XV.

APPROXIMATE SOLUTION OF EQUATIONS.

109. Real Roots of an Equation. If $f(x)$ is an integral function of x the *real* roots of the equation $f(x)=0$ can be found roughly by graphing the equation $y=f(x)$ and reading off the abscissae of the points where the graph meets the x-axis.

In Fig. 70, p. 190, is shown the graph of $y=x^2-2x-1$ for the range from $x=-1$ to $x=3$. From the graph we see that 2·41 and $-0·41$ are approximations to the roots of the equation
$$x^2-2x-1=0.$$

By now choosing larger scale units and making an entirely new graph of the equation $y=x^2-2x-1$ in the neighbourhood of $x=2·41$, we might obtain the corresponding root to more than two decimal places. A third graph with still larger scale units would lead to a still closer approximation to the root, and so on. But once a real root has been "delimited," a more expeditious method is available, which will now be explained. The method will be first applied to the solution of a quadratic equation, so that the student may have the whole process under control, the ordinary method of solving a quadratic equation and the graphing of a quadratic function being quite familiar.

110. Approximate Solution of a Quadratic Equation. Let the equation be
$$x^2-2x-1=0.$$

Denote the function x^2-2x-1 by $f(x)$ and graph the equation $y=f(x)$ (Fig. 70, p. 190). A real root of the equation $f(x)=0$ is seen to lie between 2·4 and 2·5; in

technical language one root has been "delimited." We want to find a closer approximation to this root.

The gradient of the graph is $2x-2$; near $x=2{\cdot}4$ the gradient is positive and increases with x. Let AB (Fig. 100) represent the graph from $x=2{\cdot}4$ to $x=2{\cdot}5$; the origin O is not shown on the diagram. M is the projection of A, and N that of B on the x-axis; P is the point where the arc AB and Q the point where the chord AB crosses the x-axis. OP represents the exact value of the root we are seeking and OQ an approximation to it; we have now to calculate OQ.

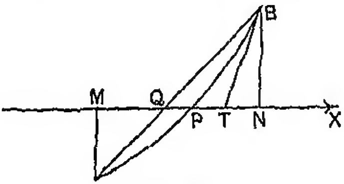


FIG. 100.

Let $MN=h$, $MA=-a'$, $NB=b'$, the numbers h, a', b' being all positive (the ordinate MA is negative). Now the triangles AMQ, BNQ are similar, and therefore

$$\frac{MQ}{a'}=\frac{QN}{b'}.$$

But each fraction is equal to $\dfrac{MQ+QN}{a'+b'}$, that is, $\dfrac{h}{a'+b'}$.

Therefore $$MQ=\frac{a'h}{a'+b'}.$$

We have also

$$OM=2{\cdot}4,\quad MA=-a'=f(2{\cdot}4)=-0{\cdot}04,$$
$$ON=2{\cdot}5,\quad NB=\ \ b'=f(2{\cdot}5)=+0{\cdot}25,$$
$$MN=ON-OM=0{\cdot}1,$$

so that $$h=0{\cdot}1,\quad a'=0{\cdot}04,\quad b'=0{\cdot}25,$$

and therefore $$MQ=\frac{0{\cdot}04\times0{\cdot}1}{0{\cdot}29}=0{\cdot}014,$$

$$OQ=OM+MQ=2{\cdot}414.$$

In obtaining this approximation we take the point Q at which the chord AB crosses the x-axis as approximately the point at which the arc AB crosses; hence the name

of the chord rule by which this method of approximation is known. The substitution of the chord AB for the arc AB enables us to calculate MQ by means of the proportion

$$MQ : MN = a' : a' + b',$$

and this method of calculating MQ is spoken of as the **Rule of Proportional Parts**—a rule that is extensively used in connection with all mathematical tables. The rule was also frequently spoken of by older mathematicians as the *Regula Falsi*, or the **Rule of Falsehood**, or the **Rule of False Position.**

We can now go on to closer approximations, by taking 2·414 and 2·415, instead of 2·4 and 2·5. We have

$$f(2{\cdot}414) = -0{\cdot}000604, \quad f(2{\cdot}415) = +0{\cdot}002225,$$

so that the curve crosses the x-axis between the points for which $x = 2{\cdot}414$ and $x = 2{\cdot}415$. We now take

$$OM = 2{\cdot}414, \quad MA = f(2{\cdot}414) = -0{\cdot}000604,$$
$$ON = 2{\cdot}415, \quad NB = f(2{\cdot}415) = +0{\cdot}002225,$$
$$MN = ON - OM = 0{\cdot}001,$$

so that now

$$h = 0{\cdot}001, \quad a' = 0{\cdot}000604, \quad b' = 0{\cdot}002225.$$

Putting these numbers in the formula for MQ, we get

$$MQ = \frac{a'h}{a' + b'} = 0{\cdot}0002135,$$
$$OQ = OM + MQ = 2{\cdot}4142135.$$

When $x = 2{\cdot}4142135$ we find by calculating $f(x)$ that $f(x)$ is negative; when $x = 2{\cdot}4142136$ it will be found that $f(x)$ is positive. We have therefore found the root with an error that is less than one unit in the 7th decimal place. The result can be confirmed by solving the quadratic in the ordinary way.

We could now proceed to a closer approximation if that were wanted.

Ex. Find correct to 4 decimal places the real roots of the following equations:

(i) $x^2 - 2x - 2 = 0$; (ii) $7x^2 + 5x - 1 = 0$; (iii) $4x^2 - 9x + 3 = 0$

by the use of graphs and the chord rule, and verify the results by solving the equations algebraically.

111. Use of two Graphs. In delimiting the real roots of an equation it is often advisable to use two graphs.

For example, let us try to find the number of real roots of the equation

$$x^3-x^2+x-2=0, \qquad \text{.......................(1)}$$

and to obtain rough approximations to their values.

Write the equation in the form

$$x^3-x^2=-x+2,$$

and then graph the equations

$$y_1=x^3-x^2, \quad y_2=-x+2$$

with reference to the same axes and with the same scale units for the two curves (Fig. 101). The graphs have only

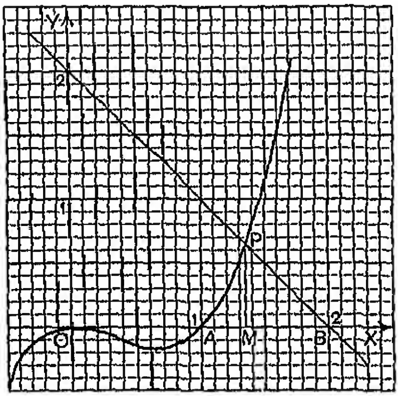


FIG. 101.

one point, P, in common, and for that point $y_1=y_2$. If a is equal to OM, the abscissa of P, then

$MP=y_1=a^3-a^2$, because P is on the graph of y_1;

$MP=y_2=-a+2$, because P is on the graph of y_2.

Therefore

$$a^3-a^2=-a+2 \quad \text{or} \quad a^3-a^2+a-2=0,$$

that is, a is a root of equation (1).

Again, the graph of y_1 crosses the x-axis at A, where $x=1$, and the graph of y_2 crosses at B, where $x=2$. The root a thus lies between 1 and 2.

We have thus shown that equation (1) has only one real root, and we have delimited the root.

Ex. 1. Prove that the equation

$$x^3-x^2+2x-3=0$$

has only one real root, and that it lies between 1 and 2.

Ex. 2. Prove that the following cubic equations have only one real root, and delimit the root:

(i) $x^3+x-3=0$; (ii) $x^3-2x-5=0$; (iii) $x^3+x^2+1=0$;
(iv) $x^3+x^2-1=0$; (v) $x^3-2x^2+5x-6=0$.

Ex. 3. Find the number of real roots of the following equations, and delimit each root:

(i) $x^3-3x-5=0$; (ii) $10x^3-10x^2+1=0$; (iii) $x^3-3x^2-x+1=0$;
(iv) $x^4-x^2+x-2=0$.

112. Combination of two Graphs and the Chord Rule. Having delimited a root by the use of two graphs, or by any other method, we can apply the chord rule to find closer approximations. If the equation to be solved is $f(x)=0$, we first find two numbers, a and b say, between which a root lies; the expressions $f(a)$ and $f(b)$ will have opposite signs.

Take the equation

$$f(x)=x^3-x^2+x-2=0.$$

We have seen (§ 111) that this equation has only one real root, and that it lies between 1 and 2. Now the gradient of the graph of $f(x)$ is given by

$$f'(x)=3x^2-2x+1.$$

When $x=1$, $f'(x)=2$, and when $x=2$, $f'(x)=9$; as x increases from 1 to 2 the gradient increases steadily from 2 to 9, so that the curve rises pretty rapidly. Before applying the chord rule we try to obtain a closer delimitation of the root.

We find, by trial, that $f(x)=-0{\cdot}103$ when $x=1{\cdot}3$, and $f(x)=+0{\cdot}184$ when $x=1{\cdot}4$, so that the root lies between 1·3 and 1·4.

The figure and relations of § 110 will apply here.

$$OM=1{\cdot}3,\quad MA=-a'=f(1{\cdot}3)=-0{\cdot}193,$$

$$ON=1{\cdot}4,\quad NB=\quad b'=f(1{\cdot}4)=+0{\cdot}184,$$

$$MN=ON-OM=0{\cdot}1,$$

so that $$h=0{\cdot}1,\quad a'=0{\cdot}193,\quad b'=0{\cdot}184,$$

and therefore $$MQ=\frac{a'h}{a'+b'}=0{\cdot}051,$$

$$OQ=OM+MQ=1{\cdot}351.$$

To test this approximation, as well as to prepare for a closer approximation, we calculate the value of $f(x)$ for $x=1{\cdot}351$ and $x=1{\cdot}352$. We find, to 4 decimal places,

$$f(1{\cdot}351)=-0{\cdot}0084,\quad f(1{\cdot}352)=-0{\cdot}0046.$$

Both 1·351 and 1·352 are too small, and we must go on calculating $f(x)$ till we find a positive value.

$$f(1{\cdot}353)=-0{\cdot}0008,\quad f(1{\cdot}354)=+0{\cdot}0030,$$

so that the root is 1·353, correct to the third decimal place.

We might now take the values

$$h=0{\cdot}001,\quad a'=0{\cdot}0008,\quad b'=0{\cdot}0030,$$

and calculate the new value of MQ. It will be found that the root lies between 1·35320 and 1·35321.

113. The Tangent Rule or Newton's Rule. There is another rule which is so generally useful for the solution of equations, whether algebraic or transcendental, that we shall give it here; it was stated by Newton.

In Fig. 100, if we draw the tangent BT at B *it will fall between the ordinate NB and the curve PB*, and if T is the point where BT crosses the x-axis, OT will obviously be a better approximation to OP than ON is. If then ON be taken as an approximation, we take OT as the next better approximation, and we shall now calculate OT.

ve $$\frac{NB}{TN}=\text{gradient at } B=m, \text{ say};$$

e $$TN=\frac{NB}{m}=\frac{b'}{m}$$

$$OT=ON-TN=ON-b'/m.$$

this to the equation of § 112, taking $ON=1{\cdot}4$ and 4. We must calculate the gradient at B.

$$f(x)=x^3-x^2+x-2, \quad f'(x)=3x^2-2x+1;$$

e (§ 103) the gradient at B is $f'(1{\cdot}4)=4{\cdot}08$.

ow find

$$OT=ON-\frac{b'}{m}=1{\cdot}4-\frac{0{\cdot}184}{4{\cdot}08}=1{\cdot}355.$$

ow begin over again, taking $1{\cdot}355$ instead of $1{\cdot}4$ value of ON and $f(1{\cdot}355)$ or $0{\cdot}0068$ as the value he new value of m may be taken as $3{\cdot}8$, and we get

$$OT=ON-\frac{b'}{m}=1{\cdot}355-\frac{0{\cdot}0068}{3{\cdot}8}=1{\cdot}3532.$$

l be found that $f(1{\cdot}3533)$ is positive and $f(1{\cdot}3532)$ is . If we go on to a further approximation we ce $1{\cdot}3533$ to be the value of ON so that B may be ie axis and BT may fall between NB and the curve.

General Statement of Rules. We shall now state the nd tangent rules in general terms.

; *Rule.* Let a real root of the equation $f(x)=0$ lie a and b, the numbers $f(a)$ and $f(b)$ being therefore ite signs; in the diagram (Fig. 100) we have

a, $MA=f(a)$; $ON=b$, $NB=f(b)$; $MN=b-a$.

quation of the chord AB is

$$y-f(a)=\frac{f(b)-f(a)}{b-a}(x-a).$$

$y=0$ we have $x=OQ$; therefore

$$OQ=a-\frac{f(a)}{f(b)-f(a)}(b-a).$$

If a is the approximation we begin with, then the term

$$\frac{-f(a)}{f(b)-f(a)}(b-a)$$

is the correction which we add to a to get the next approximation.

The value for OQ may also be written

$$OQ=b-\frac{f(b)}{f(b)-f(a)}(b-a),$$

and if b is the approximation we begin with, then the term

$$\frac{-f(b)}{f(b)-f(a)}(b-a)$$

is the correction which we add to b to get the next approximation.

Since $f(a)$ and $f(b)$ are of opposite signs, one of the corrections is positive and the other negative; it is a mere matter of convenience which of the formulae for OQ we take.

Tangent Rule. In Fig. 100 the tangent BT' falls between the curve and the ordinate at B, and we are thus certain that T is nearer to P than N is; if we draw the tangent at A, and if that tangent crosses the x-axis at T', we cannot be certain that T' will be nearer to P than M is. But the tangent rule depends only on the abscissa, the ordinate and the gradient *at one end* of the arc AB; the abscissa is the first approximation that we start from.

Attention to the following statements will lead in all cases to the choice of the end of the arc that will give the correct approximations.

(i) $f(a)$ and $f(b)$ must be of opposite signs.

(ii) $f'(x)$ must not vanish as x varies from a to b.

(iii) $f''(x)$ must not vanish as x varies from a to b.

It will be a good exercise for the student to show that condition (i) secures that there is one root between a and b, and that condition (ii) secures that there is *only one*. The third condition secures that there is no point of inflexion between A and B.

Now *let B be that end of the arc at which $f(x)$ and $f''(x)$ have the same sign*; then the tangent will fall between the ordinate NB and the curve AB, and we shall have, in the notation of § 113,

$$OT = ON - \frac{\text{ordinate at } B}{\text{gradient at } B} = b - \frac{f(b)}{f'(b)}.$$

The equation of the tangent at B is

$$y - f(b) = (x - b)f'(b),$$

and when $y=0$, $x=OT$; this gives another proof of the above value of OT.

In applying the rule we must verify at every stage that $f(b)$ and $f''(b)$ have the same sign; $f''(x)$ must not change sign, by (iii) above; and therefore if the ordinate at B is positive to begin with, it must at each subsequent stage be positive; if negative to begin with, then always negative.

EXERCISES XXIX.

Find to 3 or 4 significant figures the real roots of the equations 1 – 8.

1. $x^3 - 2x - 5 = 0.$ 2. $x^3 + x - 3 = 0.$ 3. $2x^3 + 6x - 3 = 0.$

4. $3x^3 - 4x - 5 = 0.$ 5. $x^3 - x^2 + 2x - 3 = 0.$ 6. $x^3 + x^2 - 1 = 0.$

7. $x^4 - x^3 + x - 2 = 0.$ 8. $2x^4 - 3x - 4 = 0.$

9. Calculate the root of the equation

$$x^4 - 4x^3 - 4x^2 + 10x + 10 = 0$$

that lies between 2 and 3.

10. Calculate the root of the following equation that lies between 2 and 3:

$$x^4 - 12x^2 + 12x - 3 = 0.$$

11. A sphere of radius unity is divided by a plane into two parts whose volumes are in the ratio of 1 to 2. Show that the distance x of the plane from the centre of the sphere is a root of the equation

$$3x^3 - 9x + 2 = 0,$$

and find x.

12. A hemisphere of radius unity is divided into two equal parts by a plane parallel to the base. Show that the distance x of the plane from the base is a root of the equation

$$x^3 - 3x + 1 = 0,$$

and find x.

CHAPTER XVI.

ASYMPTOTES.

115. Division by Zero. To divide a number a by a number x is to find a third number which, when multiplied by x, will give a. If, however, x happens to be zero, there is no such third number unless a is *also zero*. The working rules of algebra are carried out on the assumption that the product of two numbers is zero when one of them is zero. If then x is zero, the product of x and any other number is zero, so that if a is not zero there is *no number* which when multiplied by x will give a, and therefore there is no answer to the question, "What is the quotient of a by zero?" If, however, a is itself zero and x also zero, then any number whatever will, when multiplied by x, give a; in this case there is no *definite* answer, and the symbol $0 \div 0$ has really no meaning at all. It is perhaps worth noticing that even if we assume, for the sake of argument, that the symbol $0 \div 0$ can have a definite numerical value, we should land ourselves in all sorts of absurdities. For example,

$$8 \times 0 = 0 \quad \text{and} \quad 9 \times 0 = 0;$$

therefore $$8 \times 0 = 9 \times 0;$$

therefore $$8 \times 0 \div 0 = 9 \times 0 \div 0;$$

therefore $$8 \times (0 \div 0) = 9 \times (0 \div 0).$$

Now divide by the "number" $0 \div 0$, and we find that we have proved that 8 is equal to 9.

We have therefore to exclude division by zero from the algebraic operations. It is possible, however, in certain cases to give a useful *interpretation* of a quotient, which in

the course of an investigation is in general quite definite, but for some particular relation of the variables of the problem assumes the form $a \div 0$.

In preceding sections (*e.g.* § 88) we have tacitly assumed that the form $1 \div 0$ means "infinity," and have used the symbol ∞ for infinity; the circumstances in which this symbol was used showed clearly enough its graphical interpretation, and that was all we were concerned with. In all the cases the process was essentially that of allowing the denominator x of a fraction such as $1/x$ to become smaller and smaller, tending to zero. As x gets less and less, $1/x$ gets greater and greater, and the corresponding point on the curve goes further and further off; we say that when $x=0$ the point is "at infinity," and we then say that the symbol $1 \div 0$ represents the "number" infinity. But this "number" is not a number in the same sense that 2 is a number, any more than "infinity" is a point in the same sense that A in Fig. 77 is a point. The circumstances in which the symbol $a \div 0$ arises are essential to the whole matter, and we now give some illustrations of the utility of this "ideal number" and of the way in which it arises in investigations.

116. Infinite Root of a Simple Equation. Let B (Fig. 102) be the point $(0, b)$ referred to rectangular axes $X'OX$, $Y'OY$; the number b is supposed to be not zero. Through B draw the straight line of gradient a to meet $X'OX$ at P.

The equation of BP is

$$y = ax + b. \quad \ldots\ldots\ldots(1)$$

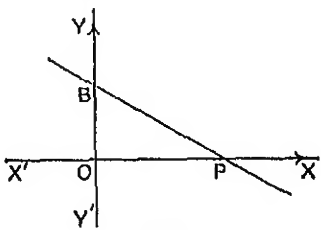


FIG. 102.

To find OP, we put $y=0$ and solve the resulting equation for x;

thus $$ax + b = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

which gives $$x = -\frac{b}{a}, \quad OP = -\frac{b}{a},$$

provided a is not zero.

Now, as a gets smaller and smaller, OP gets larger and larger; the line BP turns about B and the point P move off, say to the right, along $X'OX$. When a is very sma x is very large and P is very far off.

When $a=0$, equations (1) and (2) take the forms

$$y=0.x+b \ldots\ldots (1'), \qquad 0.x+b=0 \ldots\ldots (2').$$

Since b is not zero, equation (2′) has no solution; b the fact that equation (2′) has no solution correspon with the geometry of the case because, since $a=0$, t line BP is now parallel to $X'OX$, and therefore does r meet it.

We may now, however, *as a convenient form of spee* say that (2′) *has* a root, namely ∞, and, corresponding that BP, when it is parallel to $X'OX$, *does* meet it, at any ordinary point, but "at infinity." To say "t straight lines meet at infinity" *means exactly the sa thing* as to say "the two straight lines are parallel."

As an example, consider Fig. 33, p. 90. We proved t the line $A'C'B'D'$ is cut by the rays OA, OB, OC, OD of harmonic pencil $O(ABCD)$, so that $(A'B'C'D')$ is a harm range; we may therefore write

$$\frac{A'C'}{C'B'}=-\frac{A'D'}{D'B'}=\frac{A'D'}{B'D'} \ldots\ldots\ldots\ldots\ldots\ldots$$

But $A'D'=A'B'+B'D'$, so that

$$\frac{A'D'}{B'D'}=\frac{A'B'}{B'D'}+1. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Now turn $A'B'$ about C' till it is nearly parallel to the ratio $A'B'/B'D'$ is now very small, so that, by $A'D'/B'D'$ is nearly equal to 1, and therefore, by (3), $A'C$ is also nearly equal to 1. When $A'B'$ is exactly paral OD, $A'B'$ and OD "meet at infinity"; D' is now called point at infinity" on $A'B'$, the ratio $\dfrac{A'D'}{B'D'}$ or, as it is times written, $\dfrac{A'\infty}{B'\infty}$ is exactly equal to 1, and therefo ratio $A'C'/C'B'$ is also equal to 1, so that C' is the 1

point of $A'B'$. In fact, $A'B'$ is now $X'Y'$, which (§ 45) is bisected at C'. It is convenient to use the phrase "$(X'Y'C'\infty)$ is a harmonic range."

If the student goes back to § 4 he will see that the position-ratio AP/PB of a point P with respect to the base points A, B is never equal to -1 for an *actual* point, but continually approaches -1 as P gets further and further away from A and B; -1 is the value of $A\infty/\infty B$, or (as above) $A\infty/B\infty$ is equal to $+1$.

If C is a point that is not on the line AB, and we speak of the line joining C to "the point at infinity" on AB, then we mean the line through C parallel to AB; every other straight line through C meets AB in an actual point. Hence it follows that any number of parallel straight lines may be spoken of as "intersecting" or "meeting" at infinity; a system of parallel straight lines is a system of concurrent lines, the point of concurrence being the point at infinity on each line.

Though this mode of speech may seem strange, the student should practise it; he will soon become convinced of its advantages and will see that it involves no contradiction with the ordinary propositions of geometry. He must, however, always remember that the point at infinity on a straight line is an "ideal" point, just as infinity is an "ideal" number. Further, we must assume that there is only one point at infinity on a straight line and not two. BP (Fig. 102) can be turned so as to be parallel to $X'OX$, whether P move along OX or along OX'. When BP is not parallel, it meets $X'OX$ in only one point, and we *must assume* that when it is exactly parallel it still "meets" $X'OX$ at only one point.

117. Infinite Root of a Quadratic Equation. Let a quadratic equation be written in the standard form

$$ax^2+bx+c=0;$$

then $$x=\frac{-b+\sqrt{(b^2-4ac)}}{2a} \text{ or } \frac{-b-\sqrt{(b^2-4ac)}}{2a},$$

provided a is not zero. Let us transform the expression

for the first root so as to see its behaviour when a is supposed to be very small. We have

$$\frac{-b+\sqrt{(b^2-4ac)}}{2a}=\frac{[-b+\sqrt{(b^2-4ac)}][-b-\sqrt{(b^2-4ac)}]}{2a[-b-\sqrt{(b^2-4ac)}]}$$

$$=\frac{2c}{-b-\sqrt{(b^2-4ac)}}.$$

When a tends to zero, $\sqrt{(b^2-4ac)}$ tends to $\sqrt{(b^2)}$ or b, and the root tends to $-c/b$.

Again, when a tends to zero, the numerator of the second root tends to $-b-\sqrt{(b^2)}$ or $-2b$; the numerical value of this root therefore becomes greater and greater as a gets nearer and nearer to zero.

If then a is exceedingly small, one root of the quadratic equation is nearly equal to $-c/b$, and the other is exceedingly large. We are thus led to the following mode of speech.

When $a=0$, one root of the quadratic equation

$$ax^2+bx+c=0 \quad \text{......................(1)}$$

is $-c/b$ and the other root is infinite.

Of course it may be said, and said truly, that if $a=0$ equation (1) is not a quadratic, but is a simple equation, and therefore has only one root, namely $-c/b$. But the advantage of this other way of stating the matter lies in the fact that when, in treating a problem, the language of infinite roots is introduced, a quadratic equation, and not a simple equation, is the general expression of the relations implied in the problem, and the infinite root has a definite geometrical interpretation. We may say that we make use of the infinite root when a quadratic equation "is in question" or "is expected." (See § 118.)

If $a=0$ and also $b=0$, while c is not zero, then *both* roots of the quadratic equation are infinite.

118. Geometrical Illustration. Consider the graph of the equation

$$y=x+1+\frac{1}{x} \quad \text{or} \quad xy=x^2+x+1 \quad \text{...........(1)}$$

represented in Fig. 103.

Equation (1) is of the second degree in x and y. Any straight line, $y=ax+b$, meets the graph in two points, their abscissae being the roots of the quadratic equation

$$x(ax+b)=x^2+x+1. \quad\text{.......................}(2)$$

Whenever, then, we are discussing the intersections of a straight line with the graph of equation (1), a quadratic is to be expected.

For example, the straight line $y=\frac{2}{3}x+\frac{7}{3}$ meets the graph where

$$x(\tfrac{2}{3}x+\tfrac{7}{3})=x^2+x+1 \quad \text{or} \quad x^2-4x+3=0,$$

that is, where $x=1$ and $x=3$ (see dotted line in diagram).

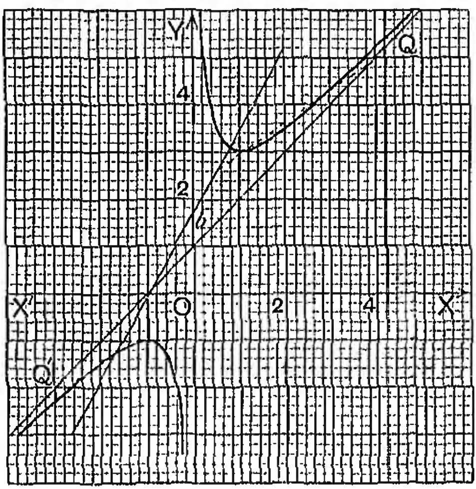


FIG. 103.

The line QQ' in Fig. 103, the equation of which is

$$y=x+1,$$

does not meet the curve at all. Solving this equation and equation (1) as simultaneous equations, we have

$$x(x+1)=x^2+x+1,$$

that is, $$0\,.\,x^2+0\,.\,x+1=0.$$

Since a quadratic equation is expected, we interpret this form of equation to mean that both roots of the quadratic are infinite.

Next take a line parallel to QQ', say $y=x+k$. Solving this equation and equation (1) as simultaneous equations, we get

$$x(x+k)=x^2+x+1 \quad \text{or} \quad 0\,.\,x^2+(k-1)x-1=0.$$

We now have one infinite root, and, so long as $k \neq 1$, one finite root, namely $1/(k-1)$. When k tends to 1, this second root also tends to infinity; the parallel to QQ' tends to become coincident with QQ', which meets the curve in "two coincident points at infinity." QQ' is an asymptote of the curve.

The y-axis is also an asymptote. To find where the y-axis meets the curve, we solve (1) and $x=0$ (that is, $x=0\,.\,y$) as simultaneous equations; the equation we get is

$$0\,.\,y^2+0\,.\,y+1=0,$$

which shows that both roots are infinite, and therefore that the y-axis meets the curve in two coincident points at infinity. All straight lines parallel to the y-axis meet the curve in one point at a finite distance and in one (ideal) point at infinity.

We remind the student of the purely conventional use of the phrase "meet at infinity"; the example we have just discussed shows how infinite roots come to be considered at all, and how it is possible to interpret them by picturing the intersections of curves as the points of intersection move off to a very great distance. We are also led to a new definition of an asymptote. Let us draw any straight line, which we may call a *search line*, say $y=ax+b$ (Fig. 103), where we suppose $b \neq 1$; this line will cut the curve (as a rule) in two distinct points. Now turn the line about the point $(0, b)$, in which it cuts the y-axis, until a is nearly equal to 1; one of the points in which it meets the curve will have gone off to a great distance, and when $a=1$, the line will be parallel to QQ', and one root of the equation

$$x(ax+b)=x^2+x+1$$

will have become infinite. Next move the search line

parallel to itself till b is nearly equal to 1; the second point of intersection is now very far off, and when $b=1$ the line coincides with QQ', and the other root of the above equation has become infinite. In other words, when our line becomes an asymptote it meets the curve in two coincident points at infinity. Hence the definition:

Definition. An asymptote to a curve is a straight line which meets the curve in two coincident points at infinity. Or, an asymptote to a curve is a tangent whose point of contact is at infinity but which is not itself at infinity.

This definition of an asymptote is not so general as that given on page 207, but it is specially suitable for curves represented by rational algebraic equations.

119. Conditions for Infinite Roots. In the equation

$$ax+b=0 \quad \text{.........................(1)}$$

put $1/z$ for x, and then multiply by z; we get

$$a+bz=0. \quad \text{.........................(1')}$$

Now when z becomes very small x becomes very large, and as z tends to zero x tends to infinity. But if $a=0$ and $b\neq 0$, equation (1′) shows that $z=0$. Hence the root of equation (1) is infinite if $a=0$ and $b\neq 0$.

Applying the same transformation to the quadratic equation

$$ax^2+bx+c=0, \quad \text{.........................(2)}$$

we get

$$a+bz+cz^2=0. \quad \text{.........................(2')}$$

One value of z is zero, and therefore one value of x is infinite, if $a=0$ and $b\neq 0$; both values of z are zero, and therefore both values of x are infinite, if $a=0$, $b=0$ and $c\neq 0$.

Quite generally, the equation

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_rx^{n-r}+\ldots+a_n=0$$

has one root infinite if $a_0=0$, $a_1\neq 0$; it has two roots infinite if $a_0=0$, $a_1=0$ and $a_2\neq 0$: it has r roots infinite if $a_0=0$, $a_1=0$, ..., $a_{r-1}=0$ and $a_r\neq 0$.

EXERCISES XXX.

Solve the simultaneous equations in Examples 1–7, stating in each case the number of points, (i) at a finite distance, (ii) at infinity, in which the graphs of the equations meet. The diagrams referred to show the graph of the second equation.

1. $y+1=0$, $y(x-1)=2-x$ (Fig. 77).

2. $y=x+1$, $y=x+1-\frac{2}{x^2}$ (Fig. 94).

3. $y=x+1$, $y=\frac{x^3-3x+2}{x^2-x+1}$ (Fig. 95).

4. $y=x+1$, $y(x-2)=x(x-1)$.

5. (a) $x=0$, $x(y-x)=1$; (b) $y=x$, $x(y-x)=1$.
   Draw the graphs.

6. (a) $x=0$, $2y=x-2+\frac{1}{x^2}$; (b) $2y=x-2$, $2y=x-2+\frac{1}{x^3}$.
   Draw the graphs.

7. (a) $x=1$, $(x-1)(y-x-1)=1$;
   (b) $y=x+1$, $(x-1)(y-x-1)=1$.
   Draw the graphs.

8. Prove that the asymptote of the curve $y=x+1+\frac{1}{(x-1)^2}$ parallel to the y-axis is the line $x=1$, and find the oblique asymptote. Graph the equation.

9. Graph the equation $x=2y-3-\frac{1}{y}$, and prove that $y=0$ and $x=2y-3$ are asymptotes.

10. Graph the following equations, and find the equations of the asymptotes

(i) $y(y-x)=1$; (ii) $y^2(y-x)=1$;
(iii) $y(y-x+1)=1$; (iv) $y^2(y-x+1)=1$.

11. Prove by Descending Continued Division that the graphs of the following equations have the asymptotes stated, and graph the equations:

(i) $y=\frac{(x-2)(x-1)}{x}$; asymptote, $y=x-3$;

(ii) $y=\frac{(x-2)(x+1)}{x-1}$; asymptote, $y=x$;

(iii) $y=\frac{(x+2)(x+4)}{x+1}$; asymptote, $y=x+5$.

d the asymptotes of the graphs of the following equations:

) $y=\dfrac{x(x+1)}{x-2}$; (ii) $y=\dfrac{x(x-1)^2}{(x-2)^2}$;

) $y=\dfrac{x^2+x+1}{x^2-x+1}$; (iv) $y=\dfrac{(x-1)(x-2)}{x^2-x+1}$;

) $y=\dfrac{x^2-x+1}{(x-1)(x-2)}$; (vi) $y=\dfrac{(x-1)(x-2)}{(x-3)(x-4)}$.

w the graph of $x=\dfrac{y^3+y-1}{y^2-y+1}$.

nt $y=x-1$ is an asymptote, and find the coordinates of the t in which the asymptote meets the curve.

'o find Asymptotes. We shall now show how, in ses, asymptotes may be found.

Inspection. Consider the equation

$$(2x-y-1)(x+2y-3)=5. \quad \text{..............(1)}$$

uation is of the second degree in x and y. Clearly ssae (or ordinates) of the points in which the line

$$2x-y-1=0$$

graph of equation (1) satisfy the equation $0=5$. quadratic equation is in question; therefore both this quadratic are infinite, and the line is an e.

dly $x+2y-3=0$ gives an asymptote.

g. 104, p. 292, for the graph.

line $ax+by+c=0$ meets a curve of the n^{th} degree curve given by an equation of the n^{th} degree in in points whose abscissae are given by an equation $(n-2)$ in x, then the line meets the curve in two infinity, and is, in general, an asymptote. (Of this statement we may replace "abscissae" by s" and x by y.) Thus

$$x+y=0, \quad x-y=0 \quad \text{and} \quad 2x-y+1=0$$

ptotes of the curve given by the equation

$$(x+y)(x-y)(2x-y+1)+5x-11y+9=0;$$

ach of the factors $x+y$, $x-y$ and $2x-y+1$, when o zero, reduces the equation from the third degree st.

II. *By Descending Continued Division.* See p. 206.

III. *By a Search Line.* Consider the equation

$$x^3+y^3-3xy=0. \qquad \qquad (1)$$

Use $y=mx+c$ as a search line. To find its intersections with the graph of (1), put $mx+c$ for y in equation (1), and arrange the resulting equation as a cubic in x; we get

$$(m^3+1)x^3+3(m^2c-m)x^2+3(mc^2-c)x+c^3=0. \qquad (2)$$

Two roots of this equation must become infinite; we therefore choose m and c to satisfy the equations

$$m^3+1=0, \quad m^2c-m=0,$$

which give $m=-1$, $c=-1$. Hence the line given by

$$y=-x-1 \quad \text{or} \quad x+y+1=0$$

is an asymptote. It will be noted that the values found for m and c make the coefficient of x in (2) also vanish, so that in this case the asymptote meets the curve in three points at infinity and nowhere else. (Fig. 105, p. 293.)

Ex. Apply this method to find the asymptotes of the cubic given to illustrate the first method.

IV. *From Freedom-Equations.* The straight line

$$ax+by=c \qquad \qquad (3)$$

meets the curve given by the freedom-equations

$$x=t^2/(t-1), \quad y=t/(t^2-1) \qquad \qquad (4)$$

in points for which the values of t are the roots of the equation

$$at^2/(t-1)+bt/(t^2-1)=c$$

or

$$at^3+(a-c)t^2+bt+c=0. \qquad \qquad (5)$$

If the line (3) is a tangent, equation (5) must have two equal roots (see Ex. 5, p. 261), and if the point of contact of the tangent is at infinity the equal roots must make one or both of the coordinates in (4) infinite.

Now we find, from (4), that (i) $x=\infty$, $y=\infty$, if $t=1$; (ii) $y=\infty$, $x=-\frac{1}{2}$, if $t=-1$; (iii) $x=\infty$, $y=0$, if $t=\infty$. We must therefore consider these three cases.

(i) If $t=1$ is a double root of (5), that equation may be written

$$(t-1)^2(at+c)=0$$

or

$$at^3+(c-2a)t^2+(a-2c)t+c=0. \quad \text{...........(6)}$$

Comparing the coefficients in equations (5) and (6), we find

$$a-c=c-2a, \quad b=a-2c,$$

and therefore $a=\frac{2}{3}c$, $b=-\frac{4}{3}c$. Equation (3) now becomes

$$\tfrac{2}{3}cx-\tfrac{4}{3}cy=c \quad \text{or} \quad 2x-4y=3,$$

and this equation gives the asymptote corresponding to $t=1$.

(ii) If $t=-1$ is a double root of (5), that equation may be written

$$(t+1)^2(at+c)=0,$$

and, comparing coefficients as before, we find $a=-2c$, $b=0$. The asymptote corresponding to $t=-1$ is therefore

$$-2cx=c \quad \text{or} \quad x=-\tfrac{1}{2}.$$

(iii) If $t=\infty$ is a double root of (5), we see that $a=0$ and $a-c=0$; that is, $a=0$, $c=0$. Hence the asymptote corresponding to $t=\infty$ is $y=0$.

The student may find the constraint equation of the curve and verify these results by the preceding methods.

121. Approach of Curve to Asymptote. To find on what side a curve approaches an asymptote, we may proceed as shown in the following examples.

Ex 1.
$$(2x-y-1)(x+2y-3)=5. \quad \text{......................(1)}$$

One asymptote is given by $2x-y-1=0$; therefore a portion, or branch, of the curve must be near this line at a great distance from the origin. We may therefore consider the equation

$$2x-y-1=0 \quad \text{or} \quad y=2x-1 \quad \text{......................(2)}$$

as the first approximation to equation (1) for points that are far off in the direction of the asymptote.

To find the second approximation, write equation (1) in the form

$$(2x-y-1)=\frac{5}{x+2y-3} \quad \text{or} \quad y=2x-1-\frac{5}{x+2y-3}. \quad \text{........(3)}$$

For points of the curve, that are far off in the direction of the asymptote we are dealing with, the value of y is equal to $(2x-1)$ approximately. Our second approximation is found by putting $2x-1$

for y in the expression on the right side of equation (3). We thus have

2nd app. $$y=2x-1-\frac{5}{x+2(2x-1)-3};$$

that is, $$y=2x-1-\frac{1}{x-1}$$

or $$y=2x-1-\frac{1}{x},$$

where $1/x$ is the most important term of the quotient $1/(x-1)$.

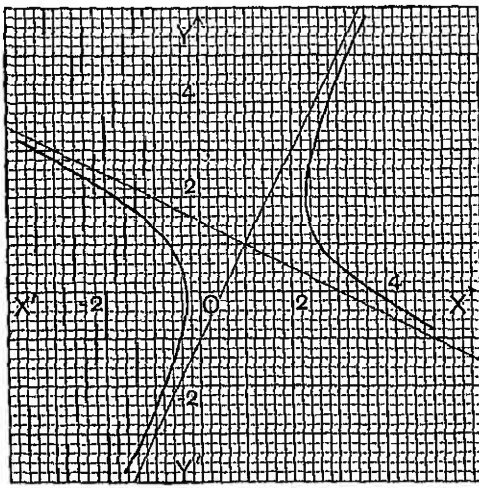


FIG. 104.

Hence the curve appears below the asymptote on the far right and above it on the far left (Fig. 104).

To find the approximation for the other asymptote, write equation (1) in the form

$$y=-\tfrac{1}{2}x+\tfrac{3}{2}+\frac{\frac{5}{2}}{2x-y-1}, \quad \text{..........................(4)}$$

and then in the expression on the right side of equation (4) put $-\frac{1}{2}x+\frac{3}{2}$ for y; that is, put for y the value in terms of x obtained from the equation of the asymptote we are now dealing with. We thus obtain

$$y=-\tfrac{1}{2}x+\tfrac{3}{2}+\frac{1}{x-1} \quad \text{or} \quad y=-\tfrac{1}{2}x+\tfrac{3}{2}+\frac{1}{x}$$

as the required second approximation. In this case the curve appears above the asymptote on the far right, and below it on the far left (Fig. 104).

Ex. 2. $$x^3+y^3-3xy=0. \quad\text{.............................(1)}$$
The asymptote is (§ 120)
$$x+y+1=0 \quad\text{or}\quad y=-x-1. \quad\text{.......................(2)}$$
At a great distance from the origin therefore in the direction given by the asymptote represented by equation (2), the curve must be close to the asymptote, and equation (2) may for such values of x and y be taken as the first approximation to equation (1).

Now write equation (1) in the form
$$x+y=\frac{3xy}{x^2-xy+y^2};$$
then
$$x+y+1=\frac{x^2+2xy+y^2}{x^2-xy+y^2}. \quad\text{..............................(3)}$$

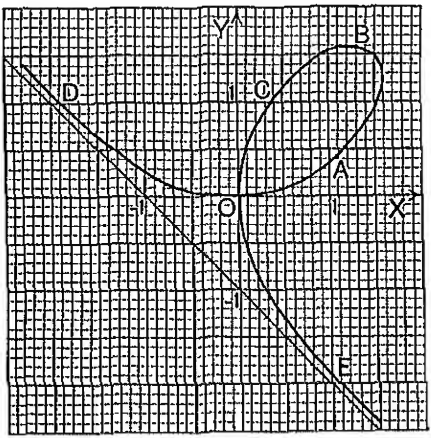


FIG. 105.

As before, in the expression on the right of equation (3), put $-x-1$ for y; that is, put the value of y in terms of x given by the first approximation (2). We then get
$$x+y+1=\frac{1}{3x^2+3x+1};$$
or, using descending division and retaining only the most important term of the quotient, namely $1/3x^2$,
$$x+y+1=\frac{1}{3x^2},$$
that is,
$$y=-x-1+\frac{1}{3x^2}.$$

Hence the curve appears above the asymptote at both "ends" of the asymptote (Fig. 105).

These examples are sufficient to indicate the methods of obtaining a knowledge of the way in which a curve approaches its asymptote. They also illustrate a method of obtaining approximations to an equation for large values of x and y. Corresponding to each asymptote that a curve has there is an approximation. In these two examples both x and y tend to infinity; we have already (§ 100) discussed the approximations when only one of the variables tends to infinity.

EXERCISES XXXI.

1. Find the asymptotes of the curves given by the following equations:

(i) $y^2-x^2=1$; (ii) $x^2-y^2=1$; (iii) $x(y-x)=1$;
(iv) $(y-x)(x+y+1)=1$; (v) $(y-2x)(x-2y)=1$;
(vi) $(2x-y+1)(x-y-2)=1$; (vii) $y(y-x)(y-2x)=1$;
(viii) $xy(y-x)(y-2x)=1$; (ix) $xy(x+y)+x^2+y^2=0$;
(x) $y^2-3x^2y+4x^4=y^4$.

2. Prove that the shape of the graph of $x^2-y^2=1$, for large values of x and y, is given by the following equations:

$$\text{(i) } y=x-\frac{1}{2x}; \qquad \text{(ii) } y=-x+\frac{1}{2x};$$

and graph the equation.

Show that the corresponding approximations for the equation $y^2-x^2=1$ are

$$\text{(i) } y=x+\frac{1}{2x}; \qquad \text{(ii) } y=-x-\frac{1}{2x};$$

and graph the equation.

3. Prove that, for large values of x and y, the approximations to the equation $(y-x)(x+y+1)=1$ are given by

$$\text{(i) } y=x+\frac{1}{2x}; \qquad \text{(ii) } y=-x-1-\frac{1}{2x};$$

and graph the equation.

Sketch the graph of $(x-y)(x+y+1)=1$.

4. Graph the equation $(y-2x)(x-2y)=1$.

5. Graph the equation $(2x-y+1)(x-y-2)=1$.

6. Show that each of the asymptotes of the curve

$$y(y-x)(y-2x)=1$$

meets the curve in three points at infinity. Prove that the curve approaches its asymptotes in the way specified by the following equations:

$$\text{Asymptote, } y=0; \qquad y=\frac{1}{2x^2}.$$

$$\text{Asymptote, } y=x; \qquad y=x-\frac{1}{x^2}.$$

$$\text{Asymptote, } y=2x; \qquad y=2x+\frac{1}{2x^2}.$$

Draw the curve.

7. Sketch the graph of the equation

$$xy(y-x)(y-2x)=1.$$

Show that the y-axis meets the curve in four points at infinity, and that the curve approaches that axis in the way specified by the equation $x=1/y^3$.

8. Trace the graph of the equation

$$xy(x+y)+x^2+y^2=0.$$

9. Draw the curves given by the following equations:

$$\text{(i) } y=3x-1+\frac{2}{(x-1)(x-2)}; \qquad \text{(ii) } y=3x-1+\frac{2}{(x-1)(x-2)^2}.$$

10. Draw the curves given by the following equations:

$$\text{(i) } y=x^2+\frac{1}{x^2}; \qquad \text{(ii) } y=(x-1)^2+\frac{1}{(x-1)^2};$$

$$\text{(iii) } y+b=(x+a)^2+\frac{1}{(x+a)^2}.$$

11. Draw the curves given by the following equations:

$$\text{(i) } y=x^2+\frac{1}{x^3}; \qquad \text{(ii) } y=x^2-\frac{1}{x^3}; \qquad \text{(iii) } y=x^3-\frac{1}{x^3}.$$

12. Graph the equation $y^2=x^2+\dfrac{1}{x^3}$.

13. Draw the curves given by the following equations:

$$\text{(i) } x(y-x)^2=1; \qquad \text{(ii) } x^2(y-x)=1; \qquad \text{(iii) } xy(y-x)=1.$$

14. Trace the curves:

$$\text{(i) } y^2(y-x)(y+2x)=1; \qquad \text{(ii) } y(y-x)^2(y+2x)=1;$$

$$\text{(iii) } y(y-x)(y+2x)^2=1.$$

15. Prove that any straight line parallel to $y+x=0$ meets the curve $x^6+y^6=x^3$ in one point at infinity, but that $y+x=0$ (the asymptote) meets the curve in three points at infinity, and that the curve appears above the asymptote at both ends.

16. Find the equation of the line that meets the curve $x^5+y^5=x^4$ in two points at infinity, and state how the curve appears at the ends of the line. Find the coordinates of the finite point in which the asymptote intersects the curve.

17. Trace the variation of the shape of the hyperbola given by the equation

$$(x-2y+1)(2x+y-1)=a,$$

as a assumes values from 1 down to zero.

What is the graph of the equation

$$(x-2y+1)(2x+y-1)=0\ ?$$

18. Factorise $2x^2-xy-y^2+x+2y-1$; then trace in one diagram the graphs of the following equations:

(i) $2x^2-xy-y^2+x+2y-2=0$; (ii) $2x^2-xy-y^2+x+2y=0$;

(iii) $2x^2-xy-y^2+x+2y-1=0$.

19. Graph in one diagram the equations:

(i) $(x+y-3)(2x-3y+4)=1$; (ii) $(x+y-3)(2x-3y+4)=0$;

(iii) $(x+y-3)(2x-3y+4)=-1$.

20. Prove that the equation

$$2x^2+3xy-2y^2-5y-3=0$$

represents a hyperbola, taking a hyperbola to mean a curve of the second degree which has two real and distinct asymptotes. Draw the curve.

21. Prove that the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents either a hyperbola or two real straight lines if $h^2>ab$, the letters other than x and y denoting constants.

22. Find how the curve given by the equations

$$x=\frac{t^2}{t-1},\quad y=\frac{t}{t^2-1}$$

approaches its asymptotes. Sketch in one diagram the curve and its asymptotes.

23. Find the equations of the asymptotes of the following curves:

(i) $x=t^2,\ y=\dfrac{t}{(t-1)^2}$; (ii) $x=\dfrac{3t}{1+t^3},\ y=\dfrac{3t^2}{1+t^3}$;

(iii) $x=\dfrac{t+1}{t-1},\ y=\dfrac{2t}{t^2-1}$.

CHAPTER XVII.

HARDER CURVES.

122. Tangent at Origin. Let the equation of a curve be written in the form

$$0 = u_1 + u_2 + u_3 + \ldots + u_n, \quad\ldots\ldots\ldots\ldots\ldots\ldots(\text{A})$$

where $u_1, u_2, u_3, \ldots u_n$ are *homogeneous* polynomials in x and y of the 1^{st}, 2^{nd}, 3^{rd}, ..., n^{th} degrees respectively; since there is no constant term the origin lies on the curve.

The equation of the tangent at the origin is $u_1 = 0$. To see this take a definite example,

$$0 = 4x - 2y + 3x^2 + 2xy - y^2 + 3x^3 - 4xy^2 + 2y^3, \quad\ldots\ldots(1)$$

so that $u_1 = 4x - 2y$, and the line to be considered is therefore

$$4x - 2y = 0. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Solving equations (1) and (2) as simultaneous equations, we get for x the equation

$$0 = 3x^2 + 3x^3 = 3x^2(1 + x). \quad\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The line (2) meets the curve (1) in two coincident points at the origin, and is therefore the tangent; any other line through the origin meets the curve in only one point there.

If equation (1) had no terms of the 2^{nd} degree, then the equation corresponding to (3) would have *three* roots equal to 0; the line (2) would therefore be an inflexional tangent. In general, if equation (A) contains u_1 and u_3 but not u_2, the origin will be a point of inflexion, because the line $u_1 = 0$ will there meet the curve in three coincident points.

Suppose now that equation (A) contains no terms of the first degree; it will then be of the form

$$0=u_2+u_3+\ldots+u_n. \quad \ldots\ldots\ldots\ldots\ldots\ldots(\text{A}')$$

In this case *every line, $y=mx$, through the origin* will meet the curve there in two coincident points, because, obviously, when we put mx for y in (A′), x^2 will be a factor of the right-hand side. The origin is therefore said to be a **double point** of the curve.

To illustrate this type take the example

$$0=ax^2+y^2-x^3. \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Here $u_2=ax^2+y^2$, and we have three cases to consider according as the factors of u_2 are (i) real and different, (ii) real and equal, (iii) imaginary.

Case (i). Factors of u_2 real and different: a negative, say $a=-1$. The equation (4) becomes

$$y^2-x^2-x^3=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(4')$$

while $u_2=(y-x)(y+x)$. The line $y-x=0$ meets the curve (4′) in *three* coincident points at the origin; similarly the line $y+x=0$ meets the curve (4′) in three coincident points at the origin. These two lines therefore lie closer to the curve than any other lines through the origin; *two* branches of the curve pass through the origin, and these lines are the tangents, one for each branch. The curve is identical with Fig. 76, p. 202, if B is taken as origin; $y-x=0$ is the tangent at B to the branch ABC, while $y+x=0$ is the tangent at B to the branch $A'BC'$. The double point is in this case a **node** (§ 83).

Case (ii). Factors of u_2 real and equal: $a=0$. The equation (4) becomes

$$y^2-x^3=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(4'')$$

while $u_2=y^2$. In this case the origin is a **cusp** (§ 83); the graph of equation (4″) is Fig. 74, p. 200.

Case (iii). Factors of u_2 imaginary: a positive, say $a=1$. The equation (4) becomes

$$y^2+x^2-x^3=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(4''')$$

while $u_2=y^2+x^2$. Here the coordinates of the origin satisfy equation (4‴), but *there is no other point of the*

curve in the neighbourhood of the origin. Writing (4‴) in the form

$$y^2 = x^2(x-1),$$

we see that, except when x and y are both zero, x must be equal to or greater than 1 if y is to be real, so that the point (1, 0) is the nearest point on the curve to the origin. The origin is called a **conjugate point** or an **isolated point.** The graph of equation (4‴) resembles Fig. 75, p. 201, *if we suppose the oval to shrink to a point at A*; it consists of the isolated point at A and an open branch DBC, where $AB=1$, the point A being the origin for the graph of equation (4‴).

If equation (A) contains no terms of the 1st and 2nd degrees and begins with u_3, then every line through the origin will meet the curve there in *three* coincident points; the origin is called a **triple point**, and the factors of u_3 furnish the tangents to the three branches that pass through the origin. Different cases arise according to the nature of the factors of u_3 (real and different, repeated, imaginary); in § 124, Ex. 4, an example of a curve with a triple point is given.

The following examples show how the gradient may be obtained in cases to which the rules of § 102 are not directly applicable.

Ex. 1. Find the gradient at any point on the graph of the equation

$$x^3+y^3-3axy=0. \quad \text{.............................(i)}$$

Let (h, k) be any point on the curve, and let the origin be shifted to the point by putting $h+\xi$ for x, and $k+\eta$ for y; equation (i) becomes

$$(h+\xi)^3+(k+\eta)^3-3a(h+\xi)(k+\eta)=0 \quad \text{.................(ii)}$$

or

$$(h^3+k^3-3ahk)+3(h^2-ak)\xi+3(k^2-ah)\eta$$
$$+3(h\xi^2-a\xi\eta+k\eta^2)+\xi^3+\eta^3=0. \quad \text{.................(iii)}$$

The term (h^3+k^3-3ahk) is zero, since (h, k) is on the curve; equation (iii) is thus of the form of equation (A) with ξ, η instead of x, y. Hence the tangent at the new origin is

$$3(h^2-ak)\xi+3(k^2-ah)\eta=0, \quad \text{.......................(iv)}$$

and therefore the gradient is

$$-\frac{3(h^2-ak)}{3(k^2-ah)}=\frac{ak-h^2}{k^2-ah}. \quad \text{..........................(v)}$$

But (h, k) is *any* point on the curve; we may therefore put x for h, and y for k, and thus get the result,

$$\text{gradient at } (x, y)=\frac{ay-x^2}{y^2-ax}.$$

In finding gradients it saves labour to write x for h and y for k in equation (ii) instead of in equation (v). We can then state the Rule:

Rule. In the equation of the curve put $x+\xi$ for x and $y+\eta$ for y, then pick out the terms of the first degree in ξ and η; if these terms are $\alpha\xi+\beta\eta$, where α and β will usually contain both x and y, the gradient at (x, y) is $-\alpha/\beta$.

If $f(x, y)=x^3+y^3-3axy$, then the derivative of $f(x, y)$ when x is variable and y kept constant is $3x^2-3ay$; if, however, x is kept constant and y is variable, the derivative is $3y^2-3ax$. It will be seen that if the expression $\alpha\xi+\beta\eta$ is formed as directed by the Rule, we shall have $\alpha=3x^2-3ay$ and $\beta=3y^2-3ax$. We are thus led to a convenient method of finding the gradient as will be shown in the next example.

Ex. 2. Let $f(x, y)$ be a polynomial in x and y; denote by f'_x the derivative of $f(x, y)$ when x is variable and y is kept constant, and by f'_y the derivative of $f(x, y)$ when y is variable and x is kept constant; then the gradient y' at any point (x, y) on the graph of the equation $f(x, y)=0$ is given by the equation

$$f'_x+f'_y\cdot y'=0 \quad \text{or} \quad y'=-\frac{f'_x}{f'_y}.$$

We shall prove the rule when $f(x, y)$ is the polynomial

$$a+bx+cy+dx^2+exy+gy^2+lx^3+mx^2y+nxy^2+py^3\,; \quad \text{.........(i)}$$

and it will be easily seen to hold for any polynomial.

In (i) put $x+\xi$ for x and $y+\eta$ for y; then pick out the terms of the first degree in ξ and η and arrange them in the form $\alpha\xi+\beta\eta$. The gradient will be $-\alpha/\beta$. The expression (i) becomes

$$a+b(x+\xi)+c(y+\eta)+d(x+\xi)^2+e(x+\xi)(y+\eta)+g(y+\eta)^2$$
$$+l(x+\xi)^3+m(x+\xi)^2(y+\eta)+n(x+\xi)(y+\eta)^2+p(y+\eta)^3.$$

The part of the first degree in ξ and η is

$$(b+2dx+ey+3lx^2+2mxy+ny^2)\xi$$
$$+(c+ex+2gy+mx^2+2nxy+3py^2)\eta.$$

The coefficient of ξ is f'_x, the coefficient of η is f'_y, and therefore the gradient is $-f'_x\div f'_y$.

This rule is of very general application. Thus take

$$y^2-2xy-x^2-3x+2y-5=0.$$

Denote the polynomial by $f(x, y)$; then

$$f'_x=-2y-2x-3, \quad f'_y=2y-2x+2$$

and
$$y'=-\frac{-2y-2x-3}{2y-2x+2}=\frac{2y+2x+3}{2y-2x+2}.$$

If $y=\dfrac{x^2-1}{x^2+1}$, then $(x^2+1)y-(x^2-1)=0$. Denote this polynomial by $f(x, y)$; we then have

$$f(x, y)=(x^2+1)y-(x^2-1), \quad f'_x=2xy-2x, \quad f'_y=x^2+1$$

and
$$y'=\frac{2x-2xy}{x^2+1}=\frac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2}=\frac{4x}{(x^2+1)^2}.$$

We have expressed the gradient in terms of x alone by putting for y its value $(x^2-1)/(x^2+1)$.

Ex. 3. Find the gradient in the following cases:

(i) $y^2+x^2-a^2=0$; (ii) $x^2+4xy-y^2-1=0$;
(iii) $y^2=2ax+bx^2$; (iv) $ax^2+2hxy+by^2+2gx+2fy+c=0$;
(v) $x(x^2+y^2)=a(x^2-y^2)$; (vi) $x^3+y^3=xy(x+y)$;
(vii) $(x^2+y^2)^2=4ax^2y$; (viii) $(y-x^2)^2=x^5$.

Ex. 4. Find y' and express the result in terms of x alone in the following examples:

(i) $(x+2)y=x-1$; (ii) $x^2y=1$; (iii) $x^ny=1$; (iv) $(x^3+1)y=x$;
(v) $(1+x^3)y=x^2-1$; (vi) $(1+x)y^2=x$; (vii) $(1+x)y^2=x^3$.

Ex. 5. If $y=u/v$ and *if u, v are polynomials in x alone*, then

$$y'=\frac{vu'-uv'}{v^2},$$

where u' and v' are the derivatives of u and v respectively.

Write the equation $y=u/v$ in the form $vy-u=0$, and denote $vy-u$ by $f(x, y)$; then

$$f(x, y)=vy-u, \quad f'_x=v'y-u', \quad f'_y=v$$

and
$$y'=-\frac{f'_x}{f'_y}=\frac{u'-v'y}{v}=\frac{vu'-uv'}{v^2},$$

since $y=u/v$, and therefore $u'-v'y=(vu'-uv')/v$.

If $y=\dfrac{1}{v}$ so that $u=1$, $u'=0$, then $y'=-\dfrac{v'}{v^2}$.

We have thus found a rule for obtaining the derivative of a quotient, and it contains also the rule for a product uv. Write uv as the quotient of u by $\dfrac{1}{v}$; then

$$y=\frac{u}{\dfrac{1}{v}}, \quad y'=\frac{\dfrac{1}{v}u'-u\,.\,\dfrac{-v'}{v^2}}{\dfrac{1}{v^2}}=vu'+uv'.$$

Ex. 6. Find y' in the following cases:

(i) $y=\dfrac{x^2+x+1}{x^2-x+1}$; (ii) $y=\dfrac{ax^2+bx+c}{Ax^2+Bx+C}$; (iii) $y=\sqrt{(x^2+1)}$;

(iv) $y=\dfrac{1}{\sqrt{(x^2+1)}}$; (v) $y=x+\sqrt{(x^2+1)}$; (vi) $y^2=(x+a)^2(x+b)$.

123. Orders of Small Quantities. In determining the shape of a curve near a given point on it, we retain some terms and reject others as being small in comparison with those retained; we are thus led to speak of different **orders** of small quantities.

When x is small, x^2 is small in comparison with x, and x^3 is small in comparison with x^2, because the ratio of x^2 to x and of x^3 to x^2 is the small quantity x. The quantities x^2 and x^3 are called small quantities of the *second* and *third* orders respectively, x itself being considered as the standard small quantity or the small quantity of the *first* order; ax^2 and bx^3, where a and b are constants, are also of the 2nd and 3rd orders respectively.

The following examples show how these notions of order are applied; we suppose the equation to be in the form $0=u_1+u_2+u_3+\dots$ of § 122.

Ex. 1. $$0=2x-4y-3x^2+4xy+2y^2+x^3. \quad\text{(i)}$$

The tangent at (0, 0) is $2x-4y=0$, and this gives the first approximation $y=\frac{1}{2}x$; near the origin therefore y is of the 1st order. Since y is of the first order, xy and y^2 are each of the 2nd order, so that the second approximation to equation (i) is $u_1+u_2=0$. It is, however, more convenient, as a rule, to give this approximation explicitly in terms of x; we therefore write

$$y=\tfrac{1}{2}x+\tfrac{1}{4}(-3x^2+4xy+2y^2), \quad\text{(ii)}$$

and in the terms on the right side, in place of y put $\frac{1}{2}x$, the value of y from the *first* approximation. We thus find

$$y=\tfrac{1}{2}x+\tfrac{1}{4}(-3x^2+4x\,.\,\tfrac{1}{2}x+2\,.\,\tfrac{1}{4}x^2)=\tfrac{1}{2}x-\tfrac{1}{8}x^2. \quad\text{(iii)}$$

Equation (iii) shows that near the origin the curve lies below the tangent.

Ex. 2. $$0=y-x^2+3xy+2y^2+x^3+y^3. \quad\text{(i)}$$

The tangent at (0, 0) is $y=0$. Near the origin *y is much smaller than x*; it is of a *higher order* of smallness. The equation suggests $y-x^2=0$ as the next approximation; this makes y of the second order, and therefore xy of the 3rd order, y^2 of the 4th order and y^3 of

the 6th order. Hence $y=x^2$ is the correct approximation, and near the origin the curve is approximately a parabola.

Similarly for the equation

$$0=x-y^2+3xy+2x^2+x^3+y^3, \quad \text{.......................(ii)}$$

the approximation is $x=y^2$; when we take y to be of the first order, xy is of the 3rd order, x^2 of the 4th and x^3 of the 6th. It depends on the given equation whether x or y is to be taken as the standard small quantity or the quantity which we call that of the first order.

Ex. 3. $$0=y-xy+2y^2-x^3. \quad \text{.............................(i)}$$

In this case the terms of the second degree contain y as a factor, and therefore vanish when $y=0$; in Example 2 (i), y is not a factor of the terms of the 2nd degree. The next approximation is given by $y-x^3=0$; this makes y of the 3rd order, xy of the 4th and y^2 of the 6th.

When u_1 is a factor of u_2, the second approximation is derived from $u_1+u_3=0$, and not from $u_1+u_2=0$. Take, for instance,

$$0=y-x-2x^2+xy+y^2-x^3-y^3. \quad \text{....................(ii)}$$

Here $u_2=(y-x)(y+2x)=u_1(y+2x)$. Write equation (ii) in the form

$$y-x=\frac{x^3+y^3}{1+y+2x}=x^3+y^3-\frac{(x^3+y^3)(y+2x)}{1+y+2x}.$$

Since the first approximation is $y=x$, the numerator of the fraction last written is of the 4th order, while the denominator is nearly unity; the fraction is thus of the 4th order, and the second approximation is therefore

$$y-x=x^3+y^3=2x^3.$$

The origin is a point of inflexion.

Ex. 4. $$x^3+y^3-3axy=0.$$

Near the origin, on the branch to which $y=0$ is the tangent, y must be much smaller than x; we therefore try $x^3-3axy=0$ or $3ay=x^2$ as the approximation. This makes y of the 2nd order, and therefore the rejected term y^3 of the 6th order, so that $3ay=x^2$ gives the correct approximation.

Similarly $3ax=y^2$ is the approximation when $x=0$ is the tangent; this makes the rejected term x^3 of the 6th order when y is of the first.

Ex. 5. $$y^4-x^4-4x^2y=0.$$

The approximation, when $y=0$ is the tangent, is $4y=-x^2$: this makes the rejected term y^4 of the 8th order.

Corresponding to the repeated tangent $x=0$, the approximation is given by $y^4-4x^2y=0$ or $2x=\pm y^{\frac{3}{2}}$. When y is of the first order, x is of the fractional order $\frac{3}{2}$, and the rejected term x^4 of the 6th order.

Ex. 6. Show that $2y^2+x^3=0$ is a first approximation to the equation of Example 1 *when x and y are large.*

The equation $2y^2+x^3=0$ gives $y=\pm(-\frac{1}{2}x^3)^{\frac{1}{2}}$; we may call y a *large*

quantity of order $\frac{3}{2}$ when x is the standard large quantity. The term xy is of order $\frac{5}{2}$, x^2 of order 2, y of order $\frac{3}{2}$, x of order 1; thus the two terms y^2 and x^3 are of the 3rd order, and the rest of lower order. Obviously x must be negative if y is real.

124. Curve Tracing. We shall now give some harder examples of curve tracing; the following general directions should be noted.

The usual procedure is to select some points on the curves, to obtain approximations to the equation for each selected point and draw the corresponding elements of the curve, and then to join up the elements thus found. In joining up the elements any symmetry, axial or central, will be very helpful; symmetry will also lessen the labour of calculating approximations. It will sometimes be possible to find values of one variable that make the other imaginary, and thus to determine regions through which the curve does not pass; *the student should look carefully for such regions.*

Important points to be examined are: the origin and the points where the curve crosses the axes, the points at infinity and points whose coordinates can be seen by inspection of the equation. If turning points can be found, these are very useful, but there is generally considerable difficulty in locating them exactly. Occasionally it will be necessary to solve equations, and the methods of Chapter XV will be useful. The determination of the gradient by the methods of § 122 will also be helpful in some cases.

It should be remembered, however, that all we profess to give are the leading features of the curve; accurate determination of its details is beyond our plan.

Ex. 1. $$x^4+y^4+a^2(x^2-y^2)=0.$$

The curve is symmetrical about both axes (Fig. 106); the origin is a node and the tangents there are $y=x$ and $y=-x$.

Near $y=x$, we have

$$y=x+\frac{x^4+y^4}{a^2(y+x)}=x+\frac{x^4+x^4}{a^2(x+x)}=x+\frac{x^3}{a^2}.$$

Near $y=-x$, we have $y=-x-\dfrac{x^3}{a^2}$.

The points $(0, a)$, $(0, -a)$ are on the curve. Shift the origin to $(0, a)$, and the equation becomes

$$0=2a^3\eta+a^2\xi^2+5a^2\eta^2+4a\eta^3+\xi^4+\eta^4,$$

and the shape near the new origin is given by $2a\eta+\xi^2=0$. Near $(0, -a)$ we have $2a\eta=\xi^2$.

By solving the equation for y we see that the greatest value of x is given by the equation $x^2=\frac{1}{2}(\sqrt{2}-1)a^2$, in which case $y^2=\frac{1}{2}a^2$. At the points given by these values of x and y the tangent is perpendicular to the x-axis. Since x^2 is not greater than $\frac{1}{2}(\sqrt{2}-1)a^2$ the values of y must also be finite, and the curve is a closed curve. The curve is shown in Fig. 106, where a is represented by 10 divisions on each axis.

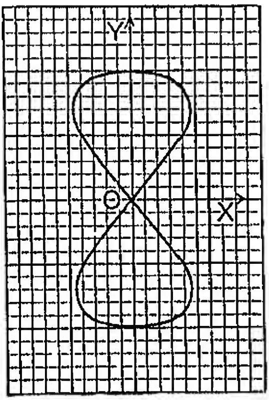


FIG. 106.

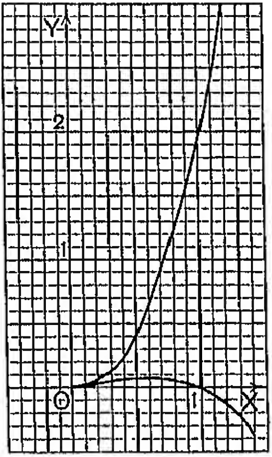


FIG. 107.

Ex. 2. $$(y-x^2)^2=x^5 \quad \text{or} \quad y=x^2\pm\sqrt{x^5}.$$

The origin is a cusp (Fig. 107), but here *both* branches lie *on the same side* of the tangent $y=0$ near the point of contact, and the cusp is called a *cusp of the second kind* (or a *rhamphoid* cusp). The lower branch crosses the x-axis at $(1, 0)$ and has a turning point where $x=0{\cdot}64$. The curve is easily traced by plotting points (Fig. 107).

Ex. 3. $$x^3+y^3-3axy=0. \qquad \text{(i)}$$

The origin is a node (Fig. 105, p. 293), and the shape there is given (§ 123, Example 4) by $3ay=x^2$ and $3ax=y^2$.

By § 122, Example 1, the gradient at any point (x, y) is $(ay-x^2)/(y^2-ax)$, and is zero when $ay=x^2$. If we solve equation (i) and the equation $ay=x^2$ as simultaneous equations, we shall find the turning points; disregarding the solution $x=0$, $y=0$, we find $x=a\sqrt[3]{2}$, $y=a\sqrt[3]{4}$ as the coordinates of a turning point. The gradient is infinite when $y^2=ax$, and by solving this equation and equation (i)

as simultaneous equations we see that the tangent at the point $(a\sqrt[3]{4},\ a\sqrt[3]{2})$ is perpendicular to the x-axis.

Since equation (i) is not altered by interchanging x and y, the curve is symmetrical about the bisector $y=x$ of the angle XOY; from this symmetry the coordinates of the point of contact of the tangent perpendicular to the x-axis might be deduced from those of the turning point.

The relation of the curve to the asymptote $x+y+a=0$ is discussed in § 121, Example 2, for the value 1 of a.

If we seek the points in which the line $y+x=\lambda$, parallel to the asymptote, meets the curve, we get the equation for x,

$$3(\lambda+a)x^2-3\lambda(\lambda+a)x+\lambda^3=0. \qquad \text{(ii)}$$

Since this equation is of the 2nd degree, one root is infinite, as it should be; the discriminant of the equation (ii) for the other two points of intersection is

$$D=9\lambda^2(\lambda+a)^2-12\lambda^3(\lambda+a)=3\lambda^2(\lambda+a)(3a-\lambda).$$

For real roots therefore we must have $3a \geqq \lambda \geqq -a$, and the curve lies between the asymptote $x+y=-a$ and the line $x+y=3a$, which touches the loop at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.

The curve is shown in Fig. 105, p. 293, for the value 1 of a.

In equation (i) put $y=tx$ and solve for x; we thus find the *freedom equations* of the curve,

$$x=\frac{3at}{1+t^3}, \quad y=\frac{3at^2}{1+t^3}, \qquad \text{(iii)}$$

and from these equations the coordinates of points may be easily calculated. It will be a good exercise for the student to show that, if P is the point given by equations (iii), the curve is traced in the following order. As t increases from $-\infty$ to -1, P moves from O along the branch OE to infinity; as t increases from -1 to 0, P returns to O from infinity along the branch DO; and finally, as t increases from 0 to $+\infty$, P describes the loop $OABC$.

Note that the line $y=tx$ meets the curve in *two* points at O and *once* at a point P different from O; the coordinates of P are therefore rational functions of t. When the freedom equations can be obtained, they enable us to calculate easily the coordinates of points, and thus to draw the curve with greater accuracy.

Ex. 4. $$y^4-x^4-4x^2y=0. \qquad \text{(i)}$$

The origin is a triple point (Fig. 108), and the approximations there are, by Example 5 of § 123,

$$4y=-x^2 \ \ldots\ldots \text{(ii)} \quad \text{and} \quad 2x=\pm\sqrt{y^3}. \ \ldots\ldots \text{(iii)}$$

This is an ordinary or *keratoid* cusp, or a cusp of the *first kind* (see Ex. 2).

The graph of (ii) is a parabola with its vertex upwards; the graph of (iii) is a semicubical parabola (§ 83) with the y-axis as the tangent.

at the origin, which is a cusp. The triple point is thus made up of an ordinary point and a cusp.

There are two asymptotes, and their relation to the curve is given by the equations

$$y=x+1-\frac{1}{2x} \quad \text{and} \quad y=-x+1+\frac{1}{2x}.$$

To find where an asymptote crosses a curve at a finite distance from the origin, we solve the equations of asymptote and curve as simultaneous equations. The asymptote $y=x+1$ crosses at the points $(-0{\cdot}3, 0{\cdot}7)$ and $(-1{\cdot}7, -0{\cdot}7)$, while the asymptote $y=-x+1$ crosses at $(0{\cdot}3, 0{\cdot}7)$ and $(1{\cdot}7, -0{\cdot}7)$; these numbers are approximate.

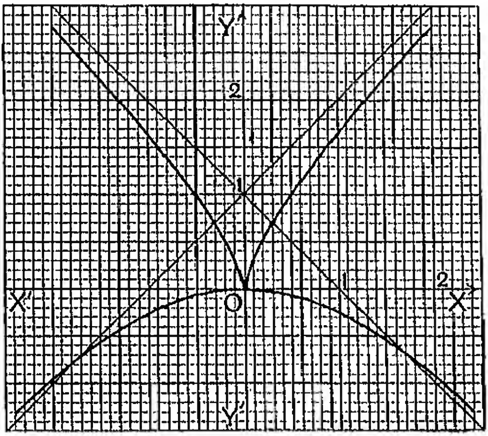


FIG. 108.

The freedom equations of the curve are

$$x=\frac{4t}{t^4-1}, \quad y=tx=\frac{4t^2}{t^4-1}.$$

It is fairly obvious now how the curve goes (Fig. 108).

Ex. 5. $$(x^2+y^2)^2=4ax^2y. \qquad \text{(i)}$$

We suppose $a>0$; we may note at once that y cannot be negative, and that the y-axis is an axis of symmetry (Fig. 109).

The origin is a triple point, and the approximations there are

$$4ay=x^2 \quad \text{and} \quad y^3=4ax^2.$$

Solving the equation for x^2, we get

$$x^2=2ay-y^2\pm 2y\sqrt{(a^2-ay)}. \qquad \text{(ii)}$$

Equation (ii) shows that y cannot be greater than a; therefore x also is finite, and the curve does not go off to infinity, as is otherwise obvious, since $(x^2+y^2)^2$ cannot vanish for real values of x and y.

From (ii) we see that two values of x^2 are equal when $y=a$; the two points (a, a), $(-a, a)$ are therefore turning points.

The freedom equations of the curve are

$$x=\frac{4at}{(1+t^2)^2}, \quad y=tx=\frac{4at^2}{(1+t^2)^2}.$$

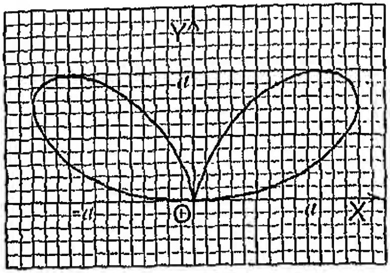


FIG. 109.

It may be shown that the gradient y' is given by the equation

$$y'=\frac{2axy-x(x^2+y^2)}{y(x^2+y^2)-ax^2}=\frac{2y(x^2-y^2)}{x(x^2-3y^2)},$$

the second fraction being obtained from the first by using equation (i). To find the points where the tangent is perpendicular to the x-axis, solve equation (i) and $x(x^2-3y^2)=0$; we get

$$x=\pm\frac{3\sqrt{3}}{4}a, \quad y=\tfrac{3}{4}a \text{ (and also } x=0,\ y=0).$$

The curve is shown in Fig. 109.

Ex. 6. $$0=2x-4y-3x^2+4xy+2y^2+x^3. \qquad \text{.......................(i)}$$

This example is much harder than the previous ones.

The curve (Fig. 110) goes through the origin and crosses the axes also at the points (1, 0), (2, 0), (0, 2). The shape near these points is given as follows:

Near (0, 0); $y=\frac{1}{2}x-\frac{1}{8}x^2$.

Near (1, 0); $2\eta^2=\xi$.

Near (2, 0); $\eta=-\frac{1}{2}\xi-\frac{3}{8}\xi^2$.

Near (0, 2); $\eta=-\frac{5}{2}\xi+\frac{1}{8}\xi^2$.

Now solve equation (i) for y in terms of x; we get

$$\begin{aligned}(2y+2x-2)^2&=(2x-2)^2-2(x^3-3x^2+2x)\\&=-2(x-1)(x^2-4x+2)\\&=-2(x-0\cdot6)(x-1)(x-3\cdot4), \qquad \text{...............(ii)}\end{aligned}$$

where we have taken $\sqrt{2}=1\cdot4$ in finding the factors of x^2-4x+2.

From equation (ii) we see that y is imaginary (a) if $x > 3{\cdot}4$, (b) if $1 > x > 0{\cdot}6$, and is real for all other values of x. When $x = 3{\cdot}4$, $v = -2{\cdot}4$ twice; when $x = 1$, $y = 0$ twice; when $x = 0{\cdot}6$, $y = 0{\cdot}4$ twice; the tangents at $(3{\cdot}4, -2{\cdot}4)$, $(1, 0)$ and $(0{\cdot}6, 0{\cdot}4)$ are perpendicular to the x-axis.

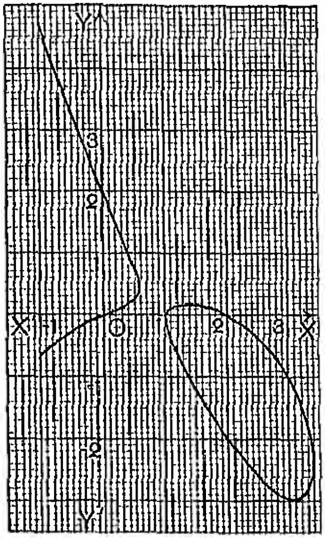


Fig. 110.

The shape near (1, 0) has already been determined; it is easy to see how the curve runs near $(0{\cdot}6, 0{\cdot}4)$ and $(3{\cdot}4, -2{\cdot}4)$, since the curve exists only on one side of the tangent at the point.

For large values of x and y we have seen in Example 6, § 123, that the curve approximates to the graph of $2y^2 = -x^3$.

The graph therefore consists of an oval and of a branch extending to infinity on both sides of the x-axis. (Fig. 110.)

Ex. 7. $$0 = y - x^2 + 3xy + 2y^2 + x^3 + y^3. \qquad \text{(i)}$$

The curve (Fig. 111) goes through the origin and meets the axes also at (1, 0) and (0, −1). The approximations are:

Near (0, 0), $y = x^2$; near (1, 0), $\eta = -\frac{1}{4}\xi - \frac{11}{32}\xi^2$; near (0, −1), $\eta^2 = -3\xi$.

The asymptote is $x + y = \frac{2}{3}$, and the relation of curve to asymptote is given by

$$y = -x + \tfrac{2}{3} + \frac{1}{x}.$$

The asymptote crosses the curve at the point $(\frac{50}{81}, \frac{4}{81})$ or $(0{\cdot}6, 0{\cdot}05)$.

To get further information, find where the curve meets the line $x+y=\lambda$ parallel to the asymptote; the abscissae of the points of intersection are given by

$$(3\lambda-2)x^2-(3\lambda^2+\lambda+1)x+\lambda(\lambda+1)^2=0. \qquad \text{(ii)}$$

Equation (ii) has two finite roots unless $\lambda=2/3$, as it should have. The discriminant D of equation (ii) is

$$D=(3\lambda^2+\lambda+1)^2-4\lambda(\lambda+1)^2(3\lambda-2)$$
$$=-3\lambda^4-10\lambda^3+11\lambda^2+10\lambda+1.$$

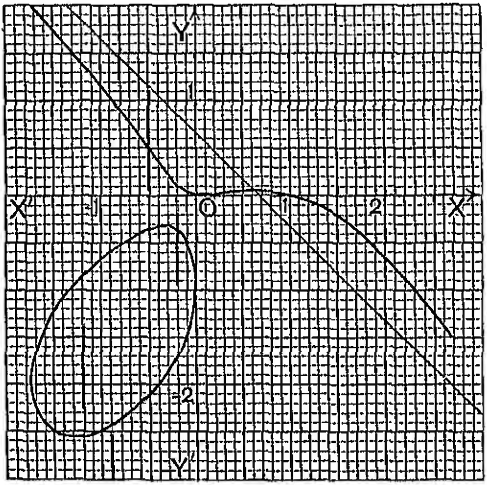


FIG. 111.

D is obviously negative for large values of λ whether λ is positive or negative. It will be found by trial that D is negative (i) if λ is greater than 1·4, (ii) if λ lies between $-0·2$ and $-0·5$, (iii) if λ is negative and numerically greater than 4·1; these values are approximate, but are sufficient to show how the curve goes.

The curve consists of a branch extending to infinity and lying between the lines $x+y=1·4$ and $x+y=-0·2$, and of an oval lying between the lines $x+y=-0·5$ and $x+y=-4·1$.

By giving to λ a series of values we can get a number of points, and thus have a fair idea of the general course of the curve. We give the table:

λ	1	0	-1	-2	-3	-4	$-0·3$
x	1, 4	0, $-0·5$	0, $-0·6$	$-0·22$, $-1·16$	$-0·69$, $-1·58$	$-1·5$, $-1·71$	$-0·08$, $-0·31$

The curve is sketched in Fig. 111.

EXERCISES XXXII.

Trace the curves given by equations 1–30:

1. $a^2(x^2-y^2)=x^4$, the Lemniscate of Gerono.
2. $a^2(x^2-y^2)=(x^2+y^2)^2$, the Lemniscate of Bernoulli.
3. $a^2x^2\pm b^2y^2=(x^2+y^2)^2$, the Lemniscate of Booth.
4. $a^2(x^2+y^2)=y^4$.
5. $a(y^2-x^2)=x(x^2+y^2)$, the Logocyclic Curve.
6. $a(y^2-x^2)=y(x^2+y^2)$.
7. $a(y^2+x^2)=x(x^2-y^2)$.
8. $a(y^2-3x^2)=x(x^2+y^2)$, the Trisectrix of Maclaurin.
9. $a(x^2+y^2)=x(3y^2-x^2)$.
10. $(a-x)y^2=x^3$, the Cissoid.
11. $ax^2+by^2=x(x^2+y^2)$, $a>0$, $b>0$.
12. $ax^2-by^2=x(x^2+y^2)$, $a>0$, $b>0$.
13. $ay^2=(x-y)(x^2+y^2)$.
14. $y^2(x+a)=x(x-a)(x-b)$, $b>a>0$.
15. $y^2(x-a)=bx(x+c)$, $a>0$, $b>0$, $c>0$.
16. $a^2y^2-b^2x^2=x^2y^2$.
17. $y(x-y)=x(x^2+y^2)$.
18. $(y-2x)^2=x^3+y^3$.
19. $a(y-x)^2=xy^2$.
20. $x^2(x+y)=(x^2+y^2)^2$.
21. $xy(x+y)=x^4+x^2y^2+y^4$.
22. $a^2xy=x^4-y^4$.
23. $a^2xy=(x^2+y^2)^2$.
24. $x(4y^2-x^2)=y^4$.
25. $x^3+y^3-x^2+3xy+4y^2+y=0$.
26. $a^2(x^2+y^2)=(x^2+y^2-ax)^2$, the Cardioid.
27. $b^2y^2=(x^2+y^2)(y-a)^2$.
28. $x^4-y^4-100a^2x^2+96b^2y^2=0$.
29. $(y-2x^2)^2=4xy^2$.
30. $y^4+x^4-5y^2-3x^2+6=0$.

31. Prove that the curve

$$8y^2=(2x+1)(4x^2-1)$$

has a double point and two real points of inflexion, and that these three points are the vertices of an equilateral triangle.

32. Trace the curve

$$(x+y+1)(2x+y+1)+x^2y=0,$$

and show that the curve

$$(x+y+1)(ax+by+c)+x^2y=0$$

will have a double point if $b=c$.

CHAPTER XVIII.

CANONICAL EQUATIONS OF THE CONIC SECTIONS.

125. The Parabola and its Canonical Equation. Let a variable point P move so that its distance from a fixed point S is equal to its perpendicular distance PM from a fixed line ZX; then the locus of P is a parabola of *focus* S and *directrix* ZX. The form of the curve is shown in Fig. 112. A, the middle point of the perpendicular SX

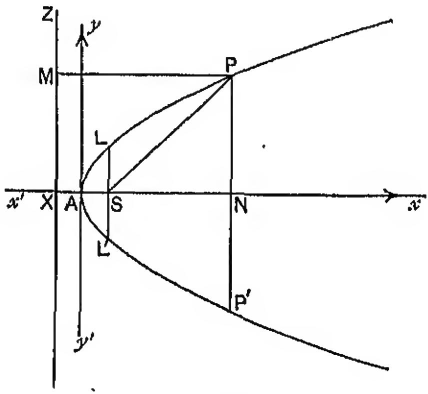


FIG. 112.

from S to ZX, is the *vertex* of the curve and the line AS is the *axis*. It is clear, from the definition, that the curve is symmetrical about its axis.

Let the curve be referred to AS and the perpendicular through A to AS as axes of x and y respectively; let AN,

NP be the abscissa and ordinate of any point $P(x, y)$ on the curve; then we have, as the equation of the curve,

$$NP^2 = 4AS \,.\, AN \quad \text{.....................(1)}$$

or

$$\mathbf{y^2 = 4ax}, \quad \text{...........................(2)}$$

where $AS = a$.

Proof.

$$\begin{aligned} NP^2 &= SP^2 - SN^2 \\ &= PM^2 - SN^2, \text{ since } SP = PM, \\ &= XN^2 - SN^2 \\ &= (AS + AN)^2 - (AN - AS)^2, \text{ since } XA = AS, \\ &= 4AS \,.\, AN \quad \text{..................................(1)} \end{aligned}$$

or

$$y^2 = 4ax. \quad \text{...(2)}$$

Since the curve is symmetrical about its axis, the y-axis is the tangent at the vertex; hence $y^2 = 4ax$ is the equation of a parabola referred to its axis and the tangent at the vertex as axes of x and y, and is called the *canonical equation* of the parabola.

Let LSL' be the double ordinate through the focus; then

$$SL = XS = 2AS = 2a,$$

therefore

$$LL' = 4a.$$

LL' or $4a$ is the *latus rectum* of the parabola.

Freedom equations of the parabola $y^2 = 4ax$ are $\mathbf{x = at^2}$, $\mathbf{y = 2at}$.

Ex. 1. Find the equation of the parabola whose focus is the origin and whose directrix is $y - x = 2$. Find also the equation of its axis, its latus rectum and the coordinates of its vertex.

(i) Let $P(x, y)$ be any point on the parabola.
Then the defining condition gives

square of distance from P to origin
= square of distance from P to $x - y + 2 = 0$

or

$$x^2 + y^2 = \frac{(x - y + 2)^2}{2};$$

whence

$$x^2 + y^2 + 2xy - 4x + 4y - 4 = 0$$

is the equation of the parabola.

(ii) The axis is the perpendicular from the focus (0, 0) to $x-y+2=0$ or

$$x+y=0.$$

(iii) The latus rectum = twice distance of focus from directrix

$$=2(2/\sqrt{2})=2\sqrt{2}.$$

(iv) With the usual notation, A, the vertex, is the middle point of SX.

Now X is the intersection of the axis and the directrix, that is, of

$$x+y=0 \quad \text{and} \quad x-y+2=0.$$

Hence X is $(-1, 1)$; therefore A is $(-1/2, 1/2)$.

Ex. 2. AQ, AR are chords of a parabola drawn at right angles to each other from the vertex A. The rectangle $AQPR$ is completed on AQ, AR. Prove that the locus of P is another parabola.

Let the given parabola be $y^2=4ax$.

Let Q, R be the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$.

Then the gradients of AQ, AR are $\dfrac{2at_1}{at_1^2}, \dfrac{2at_2}{at_2^2}$ or $\dfrac{2}{t_1}, \dfrac{2}{t_2}$.

But AQ, AR are at right angles; therefore $t_1t_2=-4$. ....(i)

The coordinates of the middle point of QR are $\dfrac{a(t_1^2+t_2^2)}{2}$, $a(t_1+t_2)$.

Hence the coordinates (h, k) of P are such that

$$h=a(t_1^2+t_2^2) \quad \text{and} \quad k=2a(t_1+t_2),$$

or

$$h=a(t_1+t_2)^2-2at_1t_2 \quad \text{.........................(ii)}$$

and

$$k=2a(t_1+t_2). \quad \text{....................................(iii)}$$

Using (i) and (iii) to substitute in (ii), we get

$$h=\frac{k^2}{4a}+8a \quad \text{or} \quad k^2=4a(h-8a).$$

Writing x for h and y for k to denote a varying point P, we have

$$y^2=4a(x-8a)$$

as the equation of the locus of P; and this represents a parabola whose vertex is the point $(8a, 0)$, whose latus rectum is $4a$ and whose concavity is in the direction of the x-axis.

Ex. 3. Prove that the chord $x+by=4a$ subtends a right angle at the vertex of the parabola $y^2=4ax$.

If we put $x+by$ for $4a$ in the equation $y^2=4ax$, we get

$$y^2=x(x+by) \quad \text{or} \quad x^2+bxy-y^2=0,$$

a homogeneous equation of the second degree which represents the two straight lines joining the origin, or vertex, to the intersections of the chord and the curve. Since the sum of the coefficients of x^2 and y^2 is zero, these lines are perpendicular, by § 42; so that the chord subtends a right angle at the vertex.

This artifice of making one equation homogeneous by means of another is of some importance; it enables us at once to get an equation giving the lines joining the origin to the intersections of the curves specified by the equations.

EXERCISES XXXIII.

1. Draw the parabola whose focus is the origin and whose directrix is $3x-4y=6$. Find its equation, the equation of its axis, its latus rectum and the coordinates of its vertex.

2. Draw the parabolas whose foci and directrices are as follows; find their equations, the equations of their axes, their latera recta, and the coordinates of their vertices:

(i)	focus: $(3, 4)$;	directrix:	$y=x$.
(ii)	„ $(2, 1)$;	„	$4x-3y-6=0$.
(iii)	„ $(1, 1)$;	„	$3x+4y-8=0$.
(iv)	„ $(1, -1)$;	„	$4x+3y-6=0$.
(v)	„ $(0, 0)$;	„	$5x+12y-13=0$.

3. Show that $y=x-x^2$ may be defined geometrically as the locus of a point which moves so that its distance from $(1/2, 0)$ is equal to its distance from $y=1/2$. Give a geometrical definition of $x=y-y^2$. Sketch both curves.

4. Find the focus and directrix of each of the parabolas

$$y=x^2,\quad y=-x^2,\quad x=y^2,\quad x=-y^2.$$

5. Refer the curve whose equation is $y=x^2-4x+1$ to parallel axes of ξ and η through the point $x=2$, $y=-3$, and prove that it represents a parabola. Find its latus rectum; and also the focus and directrix referred to the x and y axes.

6. Prove that $y=x^2$, $y=(x-1)^2$, $y=x^2-2x+2$, $y=x^2+3x+3$ and $y=x^2+px+q$ are congruent parabolas.

7. Prove that $y^2=4ax$ and $y^2=4a(x+a)$ are congruent parabolas.

8. Prove that $y=2x^2$, $y=2x^2+4x+3$, $y=2x^2+px+q$ are congruent parabolas.

9. Prove that $y=ax^2+bx+c$ represents the parabola whose vertex is $\left(-\dfrac{b}{2a}, -\dfrac{b^2-4ac}{4a}\right)$, whose latus rectum is the absolute value of $1/a$, whose axis is parallel to the y-axis, and whose concavity is upwards or downwards according as a is positive or negative.

10. Find the equation of the parabola whose axis is parallel to the y-axis and which passes through the points $(1, -3)$, $(2, -4)$, $(3, -1)$.

11. If ax^2+bx+c and dx^2+ex+f have equal values when $x=x_1$, $x=x_2$, and $x=x_3$, prove that they have equal values for all values of x.

12. Parabolas with their axes parallel to the y-axis are drawn through the following sets of three points; find whether their concavities are turned upwards or downwards:

(i) (0, 1), (−1, 6), (2, 3); (ii) (0, 1), (−2, −5), (3, −5);
(iii) (−1, 2), (−2, −2), (1, −8); (iv) (1, 0), (−1, 4), (2, 7).

13. Find the distance between the foci of each of the following pairs of parabolas:

(i) $y^2=4ax$ and $y^2=4b(x+b)$;
(ii) $y^2=4a(x+b)$ and $y^2=4a(x+d)$.

14. P is the centre of a variable circle which touches the x-axis and the fixed circle, centre $(0, a)$, radius a; prove that the locus of P is a parabola. Find its latus rectum, the coordinates of its focus and the equation of its directrix.

15. A variable circle passes through a fixed point A and touches a fixed straight line L; and AP is the diameter of the circle through A. Prove that the locus of P is a parabola; and find its focus, vertex and directrix.

16. A variable point P moves so that the length of the tangent from P to the circle $x^2+y^2=2ay$ is equal to the ordinate of P; prove that the locus of P is a parabola.

17. The vertex A of a variable triangle ABC is the fixed point $(0, p)$; the variable vertices B, C move on the x-axis so that OB^2+OC^2 is constant, where the origin O is the foot of the perpendicular from A to BC. Prove that the locus of the circum-centre of the triangle is a parabola.

18. A variable circle cuts a fixed circle at right angles and touches a fixed straight line. Prove that the locus of its centre is a parabola.

19. A chord PQ of the parabola whose vertex is A meets the axis in O. If MP, NQ are the ordinates of P and Q, prove that

$$AM \cdot AN = AO^2.$$

20. Two chords PP' and QQ' of a parabola intersect at a point on the axis; prove that the rectangle contained by the ordinates of P and P' is equal to the rectangle contained by the ordinates of Q and Q'.

21. If Pp be a focal chord of a parabola, of vertex A and focus S, and if AP, Ap meet the latus rectum in Q, q, prove that SQ, Sq are equal to the ordinates of p, P.

22. NP is the ordinate of any point P on a parabola of vertex A. PM is drawn perpendicular to AP to meet the axis in M; MQ perpendicular to the axis meets the parabola in Q. Prove that

$$QM^2-PN^2=16AS^2.$$

23. A parabola is described through four consecutive angular points of a regular hexagon. Show that the radius of the circle inscribed in the hexagon is equal to the latus rectum of the parabola.

24. If from the vertex of the parabola $y^2=4ax$ a pair of chords be drawn at right angles to each other, find the equation of the locus of the middle point of the chord joining the further extremities. Show that the locus is a parabola, and find its latus rectum and the coordinates of its vertex and focus.

25. Prove that the locus of the middle points of a series of chords of a parabola drawn through its vertex is a parabola.

26. Given the axis of a parabola and two points on the curve, determine the focus and directrix.

27. A variable chord of a parabola subtends a right angle at the vertex. Prove that it passes through a fixed point.

28. Find the coordinates of the middle point of the chord of the parabola $y^2=4ax$ whose equation is $y=x+c$, and deduce the equation of the locus of the middle point as c varies.

29. Prove that $y=2a/m$ is the locus of the middle points of the chords of the parabola $y^2=4ax$ which have a gradient m.

30. Prove that $y=mx+c$ is a tangent to the parabola $y^2=4ax$ if $c=a/m$.

126. The Ellipse. Let a variable point P move so that its distance SP from a fixed point S bears a constant ratio

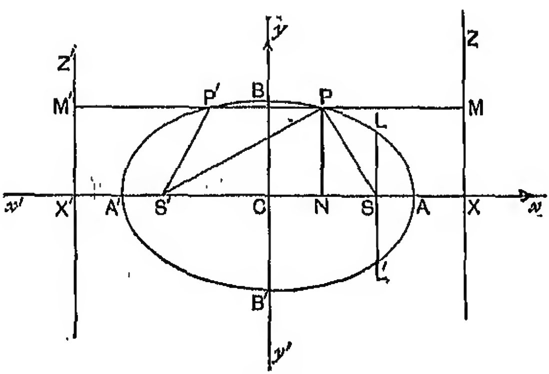


FIG. 113.

e, less than unity, to its distance PM from a fixed line ZX; then the locus of P is an ellipse of *eccentricity* e, *focus* S and *directrix* ZX. The form of the curve is shown in Fig. 113.

Let A and A' divide SX, the perpendicular from S to ZX, internally and externally, so that

$$SA = e \,.\, AX \quad \text{and} \quad SA' = e \,.\, A'X;$$

let C be the middle point of AA'; and let $B'CB$, $L'SL$, perpendicular to AA', meet the curve in B', B and L', L. By the definition of the ellipse, $SB = e \,.\, CX$ and $SL = e \,.\, SX$. Then

AA' is called the *major axis* and BB' the *minor axis*;

C is called the *centre* and LL' the *latus rectum* of the ellipse.

Let $$AA' = 2a \quad \text{and} \quad BB' = 2b.$$

THEOREMS.

$$CS = e \,.\, CA = ea; \qquad \text{.........................(1)}$$

$$CA = e \,.\, CX \quad \text{or} \quad CX = a/e; \qquad \text{..............(2)}$$

$$CS \,.\, CX = CA^2 = a^2; \qquad \text{..........................(3)}$$

$$b^2 = a^2(1 - e^2); \qquad \text{..........................(4)}$$

$$SL = b^2/a. \qquad \text{..............................(5)}$$

Proof. (1) $SA = e \,.\, AX$ and $SA' = e \,.\, A'X$;

therefore, by subtraction,

$$SA' - SA = e(A'X - AX), \text{ that is, } 2CS = 2e \,.\, CA,$$

whence $$CS = e \,.\, CA = ea.$$

(2) Also, by addition,

$$SA' + SA = e(A'X + AX), \text{ that is, } 2CA = 2e \,.\, CX$$

whence $$CA = e \,.\, CX \quad \text{or} \quad CX = a/e.$$

(3) $$CS \,.\, eCX = eCA \,.\, CA, \quad \text{by (1) and (2)};$$

hence $$CS \,.\, CX = CA^2 = a^2.$$

(4) $$\begin{aligned} b^2 = CB^2 = SB^2 - CS^2 &= (e \,.\, CX)^2 - CS^2 \\ &= a^2 - a^2e^2, \quad \text{by (1) and (2)}, \\ &= a^2(1 - e^2). \end{aligned}$$

(5) $$SL = e \,.\, SX = e(CX - CS) = e\left(\frac{a}{e} - ae\right) = a(1 - e^2) = b^2/a.$$

The student will have noticed that we have not used "steps" in the proofs. In this respect we are following the customary practice of *Geometrical Conics*, and we shall continue to do so in similar cases.

EXERCISES XXXIV.

1. If $CS=5$ and $CA=6$, calculate e and the distances of S and C from the directrix.

2. If $CA=6$ and $CX=7$, calculate e and the distances of S and A from the directrix.

3. If $a=5$ and $b=3$, calculate e and SX.

4. If $CA=6$ and $e=1/2$, calculate CB, SX and SA.

5. If $CS=4$ and $CX=6$, calculate CA, CB and e.

6. If $a=5$ and $e=4/5$, calculate b, CS, CX, AX.

7. If $b=4$ and $e=3/5$, calculate a, CS, SA, SX.

8. If $a=5$ and $b=3$, calculate the length of the latus rectum.

9. If the focus of an ellipse, whose eccentricity is 1/2, is 3 ins. from the directrix, calculate the lengths of the major and minor axes.

10. Prove that $SB=CA$ and that $CS^2+CB^2=CA^2$.

11. If $AS/A'S=29/30$, as in the case of the Earth's path round the Sun, calculate e.

12. If $CS=CB$, calculate e.

13. Prove that $AS \, . \, A'S=CB^2$.

In the following exercises the ellipse is referred to CA and CB as axes of x and y; CN, NP are the abscissa and ordinate of a point $P(x, y)$ on the curve.

14. If $a=5$, $b=3$, $SP=4$, prove that $e=4/5$, $NX=5$, $CN=5/4$, $NP=\pm 3\sqrt{15}/4$ and $x^2/a^2+y^2/b^2=1$.

15. If $a=5$, $b=3$, $SP=3$, prove that $x=5/2$, $y=\pm 3\sqrt{3}/2$ and $x^2/a^2+y^2/b^2=1$.

16. If (x, y) are the coordinates of L, prove that $x^2/a^2+y^2/b^2=1$.

17. If (x, y) is a point where $y=x$ meets the ellipse, prove that $x=\pm ab/\sqrt{a^2+b^2}=y$ and $x^2/a^2+y^2/b^2=1$.

18. P is any point on an ellipse, M its projection on the directrix. If the bisector of the angle SPM meet SM in Q, find the locus of Q.

19. If the focus of an ellipse be the common focus of two parabolas whose vertices are at the ends of the major axis, these parabolas will intersect in points whose distance from each other is equal to twice the minor axis.

127. Canonical Equation of the Ellipse. If the ellipse, whose semi-major and semi-minor axes are $CA = a$ and $CB = b$, is referred to CA and CB as axes of x and y, its equation, then called the *canonical* equation, is

$$\frac{\mathbf{x}^2}{\mathbf{a}^2}+\frac{\mathbf{y}^2}{\mathbf{b}^2}=1.$$

Let $P(x, y)$ be any point on the curve (Fig. 113); let $ON = x$, $NP = y$.

$$\begin{aligned}\text{Then}\quad y^2 = NP^2 &= SP^2 - NS^2 = e^2 . PM^2 - NS^2\\ &= e^2 . NX^2 - NS^2\\ &= e^2(CX - CN)^2 - (CS - CN)^2\\ &= e^2\left(\frac{a}{e} - x\right)^2 - (ae - x)^2, \quad \text{by (1) and (2) of § 126,}\\ &= (a - ex)^2 - (ae - x)^2\\ &= a^2 + e^2x^2 - a^2e^2 - x^2\\ &= (a^2 - x^2)(1 - e^2)\\ &= (a^2 - x^2) . \frac{b^2}{a^2}, \quad \text{by (4) of § 126.}\end{aligned}$$

Therefore $$\frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2} = 1 - \frac{x^2}{a^2}$$

or $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Note. (1) The equation may be put in the form

$$\frac{y^2}{a^2 - x^2} = \frac{b^2}{a^2} \quad \text{or} \quad \frac{NP^2}{CA^2 - CN^2} = \frac{CB^2}{CA^2},$$

or, as it is usually written,

$$\frac{NP^2}{AN . NA'} = \frac{CB^2}{CA^2}.$$

(2) The curve is symmetrical about both axes. For the image of $P(x, y)$ in AA' is $(x, -y)$, which lies on the curve; and the image of P in BB' is $(-x, y)$, which also lies on the curve.

(3) Every straight line drawn through C and terminated by the curve is bisected at C. This follows from (2). Such a line is called a *diameter*.

(4) For any point $Q(x_1, y_1)$ *inside* the curve,

$$x_1^2/a^2+y_1^2/b^2<1.$$

For if CQ meet the curve in $P(x, y)$, $x_1<x$ and $y_1<y$ (numerically), so that $x_1^2/a^2+y_1^2/b^2<x^2/a^2+y^2/b^2$, and therefore <1.

(5) For any point $R(x_2, y_2)$ *outside* the curve,

$$x_2^2/a^2+y_2^2/b^2>1.$$

If CR meet the curve in $P(x, y)$, $x_2>x$ and $y_2>y$ (numerically), so that $x_2^2/a^2+y_2^2/b^2>x^2/a^2+y^2/b^2$, and therefore >1.

Ex. 1. Find the equation of the ellipse whose focus is the origin, directrix $x+y+2=0$, and eccentricity $1/2$. Find also the coordinates of its centre.

Let S be the focus, PM the perpendicular from a point $P(x, y)$ on to the directrix; then $$SP^2=e^2PM^2$$

or $$x^2+y^2=\tfrac{1}{4}\left(\frac{x+y+2}{\sqrt{2}}\right)^2,$$

which reduces to $\quad 7x^2+7y^2-2xy-4x-4y-4=0.$

Let X be the foot of the perpendicular from S to the directrix; then X is the intersection of $x-y=0$ and $x+y+2=0$, that is, $(-1, -1)$. But $SA/AX=1/2$ and $SA'/A'X=-1/2$, where A and A' are the vertices. Hence (§ 10) A and A' are the points $(-1/3, -1/3)$ and $(1, 1)$. But C, the centre, bisects AA'; therefore C is the point $(1/3, 1/3)$.

Ex. 2. Perpendiculars through P to PA and PA' meet the major axis in M and M'; prove that MM' is equal to the latus rectum.

Let P be the point (x_1, y_1). Then gradient of PA is $y_1/(x_1-a)$; so that gradient of PM is $(a-x_1)/y_1$, and the equation of PM is

$$y-y_1=\frac{a-x_1}{y_1}(x-x_1).$$

When $y=0$, $x=CM$; therefore $CM=x_1+\dfrac{y_1^2}{x_1-a}$.

Similarly $$CM'=x_1+\frac{y_1^2}{x_1+a}.$$

Therefore $MM'=CM'-CM=2a\,.\,\dfrac{y_1^2}{a^2-x_1^2}=\dfrac{2b^2}{a}$, since $x_1^2/a^2+y_1^2/b^2=1$.

EXERCISES XXXV.

1. Find the eccentricity of the ellipse $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$.

2. Find the eccentricity and the latus rectum of the ellipse

$$x^2/25+y^2/9=1.$$

3. Find the canonical equation of the ellipse whose minor axis is 6 and latus rectum 2.

4. Find the distance from the focus to the directrix of the ellipse

$$x^2/16+y^2/7=1.$$

5. Find which of the following points are inside and which are outside the ellipse specified by $a=5$ and $b=3$:

$$(2, -3)\,;\ \ (-4, 2)\,;\ \ (-0{\cdot}3, 3{\cdot}2)\,;\ \ (3{\cdot}2, 2{\cdot}8).$$

6. Determine the points of intersection of the ellipse

$$x^2/a^2+y^2/b^2=1,$$

and the diameters formed by the bisectors of the angles formed by the axes. Find also the length of either diameter.

7. Find the distance from the centre of the ellipse specified by $a=5$, $b=3$, to either of the points on the curve whose abscissa is 2.

8. Establish the following formula for the length of the semi-diameter of gradient $\tan\theta$:

$$r^2=\frac{b^2}{1-e^2\cos^2\theta}.$$

This is the *polar equation* of the ellipse, referred to C as pole and CA as initial line.

9. If the diameter of gradient unity is equal to the major semi-axis, find the eccentricity of the ellipse.

10. Establish the formulae:

$$SP=\sqrt{\{(ae-x)^2+y^2\}}\,;$$
$$SP=a-ex.$$

11. The points (x, y), $(-x, y)$, $(-x, -y)$, $(x, -y)$, A, A' are the vertices of an equilateral hexagon inscribed in the ellipse

$$x^2/a^2+y^2/b^2=1\,;$$

prove that
$$x=\pm\frac{a(a^2+b^2)}{3a^2+b^2}.$$

12. If CA and CB are the lines $2x-y+1=0$ and $x+2y-3=0$, and $CA=5$, $CB=3$, find the equation of the ellipse.

13. If CB is the line $4x+3y+2=0$, and S and A are the points $(1, 1)$ and $(2, 7/4)$, find the equation of the ellipse.

14. Find the centres, and the equations and lengths of the axes of the following ellipses; and sketch the curves:

(i) $\dfrac{x^2}{5}+\dfrac{y^2}{7}=1.$ (ii) $\dfrac{x^2}{(a+b)^2}+\dfrac{y^2}{(a-b)^2}=1.$

(iii) $\dfrac{(x-1)^2}{4}+\dfrac{(y+2)^2}{3}=1.$ (iv) $\dfrac{(x-y)^2}{18}+\dfrac{(x+y)^2}{8}=1.$

(v) $\dfrac{(3x+4y)^2}{125}+\dfrac{(4x-3y)^2}{80}=1.$ (vi) $\dfrac{(x-2y+1)^2}{30}+\dfrac{(2x+y+1)^2}{7}=1.$

(vii) $3x^2+4y^2-6x+8y=5.$ (viii) $8x^2+45y^2-8x+60y+12=0.$

(ix) $9x^2+3y^2-12x+6y=2.$ (x) $9x^2+3y^2+12x+6y=2.$

15. Draw on squared paper the ellipse whose focus is the origin, directrix $x-y=5$ and eccentricity $4/5$. Find its equation and its latus rectum.

16. Draw on squared paper the ellipse whose focus is $(1, 2)$, directrix $3x-4y+10=0$ and eccentricity $1/2$. Find its equation and the coordinates of its centre.

17. One vertex of a variable parallelogram $OAPB$ is fixed at the origin O. The adjacent sides $OA=a$ and $OB=b$ make equal and opposite angles with the x-axis, while the parallelogram opens and shuts. Prove that P traces out the ellipse

$$\frac{x^2}{(a+b)^2}+\frac{y^2}{(a-b)^2}=1.$$

18. A straight line of fixed length moves so that its extremities lie on two rectangular axes; prove that every point on it traces out an ellipse.

19. If a circle roll inside another of double its radius, show that the extremities of a diameter of the rolling circle, invariably connected with it, describe perpendicular straight lines and that any point invariably connected with the rolling circle but not on its circumference describes an ellipse.

20. Circles are described on ACA', BCB', the axes of an ellipse, as diameters; a variable radius vector CQR meets the first circle in Q and the second in R; parallels through Q and R to CB and CA respectively meet in P. Prove that the locus of P is the ellipse.

Draw the ellipse specified by $a=5$, $b=3$.

21. P is a point on the ellipse whose axes are $A'CA$, $B'CB$ and n is the projection of P on the minor axis $B'CB$; prove that

$$nP^2 : B'n \, . \, nB = CA^2 : CB^2.$$

22. NQ is a variable ordinate of the circle $x^2+y^2=a^2$, and P is taken on NQ so that $NP=\frac{1}{2}NQ$; prove that the locus of P is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{a^2/4}=1.$$

If $NP=\dfrac{b}{a}\,.\,NQ$, find the equation of the locus of P, and the area contained by the locus.

23. ACA' and BCB' are straight lines of lengths $2a$ and $2b$ respectively, which bisect one another at right angles at C; on ACA' are taken points Q and Q' so that $CQ\,.\,CQ'=a^2$. If BQ, $B'Q'$ cut at P, find the equation of the locus of P, and sketch the locus.

24. A point P moves so that the length of the tangent from P to a given circle bears a constant ratio e, less than unity, to the perpendicular distance of P from a given tangent to the circle. Prove that the locus of P is an ellipse whose eccentricity is e and whose latus rectum is equal to the diameter of the circle.

25. CP and CQ are two perpendicular semi-diameters of the ellipse $x^2/a^2+y^2/b^2=1$; prove that

$$\frac{1}{CP^2}+\frac{1}{CQ^2}=\frac{1}{a^2}+\frac{1}{b^2}.$$

26. If the two ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{and} \quad \frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=1$$

have the same eccentricity, prove that $a/a_1=b/b_1$. If a radius vector CPQ meet the first in P and the second in Q, prove that $CP : CQ$ is constant. (The ellipses are said to be *homothetic* or *similar and similarly situated.*)

27. If P is a point on the ellipse $x^2/a^2+y^2/b^2=1$, and CS' equal to CS is cut off from CA', and if SP and $S'P$ are at right angles, show that the ordinate of P is equal in length to SX.

28. On CA, CB as diameters, circles are described; find the coordinates of their points of intersection with the ellipse whose semi-axes are CA and CB.

29. If a tangent to the circle whose equation is

$$x^2+y^2=a^2b^2/(a^2+b^2)$$

meet the ellipse $x^2/a^2+y^2/b^2=1$ in P and Q, prove that PQ will subtend a right angle at the centre of the ellipse.

30. Find the coordinates of the middle point of the chord of the ellipse $2x^2+3y^2=6$, whose equation is $x+y=1$.

31. Find the locus of the middle points of chords of the ellipse $ax^2+by^2=1$ parallel to the diameter $y=mx$.

32. Find the equation of the ellipse whose focus is (1, 2), directrix $x+y=1$, eccentricity 1/2. Find the equation of the straight line

which bisects all chords parallel to the x-axis. Draw the curve and the line.

33. Prove that a pair of common chords of the circle $x^2+y^2=a^2-b^2$ and the ellipse $x^2/a^2+y^2/b^2=1$ pass through the common centre. Find the equation of the pair.

34. Prove that $$y=mx\pm\sqrt{a^2m^2+b^2}$$ are the tangents of gradient m to the ellipse $x^2/a^2+y^2/b^2=1$, and that the join of the points of contact passes through the centre.

35. Prove that $lx+my=n$ is a tangent to the ellipse $x^2/a^2+y^2/b^2=1$ if $a^2l^2+b^2m^2=n^2$.

128. Foci of the Ellipse. If the ellipse (Fig. 113) be folded about the minor axis BB' so that A falls on A', the two halves of the figure will be superposed. Let P fall on P', S on S', ZX on $Z'X'$, PM on $P'M'$. Then $S'P'=e.P'M'$, so that the ellipse may be traced from S' and $Z'X'$ as focus and directrix, instead of from S and ZX. The ellipse has thus two foci and two corresponding directrices, and the following theorem, which was taken in § 71 as the definition of an ellipse, can now be proved.

THEOREM.

The sum of the focal distances of any point on an ellipse is constant and equal to the major axis; or

$$SP+S'P=AA'.$$

Proof. We have (Fig. 113), if $CN=x$,

$$SP=e.PM=e.(CX-CN)=e.CX-e.CN=a-ex, \quad (1)$$
$$S'P=e.M'P=e.(X'C+CN)=e.CX+e.CN=a+ex, \quad (2)$$

because $e.CX=CA=a$. By addition, we now get

$$SP+S'P=2a=AA'.$$

The relations (1) and (2) that express the focal distances of a point in terms of a, e, and x, the abscissa of the point, are of some importance.

EXERCISES XXXVI.

1. A string, 10 in. long, has its extremities at fixed points 8 in. apart. If the string is kept tight by a moving point, find (i) the eccentricity, (ii) the major axis, (iii) the minor axis of the ellipse traced out by the moving point.

2. If the ellipse $x^2/40+y^2/16=1$ is drawn by the use of a string, find the length of the string and the distance apart of its extremities.

3. If S and S' are the points $(ae, 0)$, $(-ae, 0)$, and a variable point $P(x, y)$ moves so that $SP+S'P=2a$, establish directly the equations:

(i) $SP=\sqrt{(ae-x)^2+y^2}$, (ii) $S'P=\sqrt{(ae+x)^2+y^2}$, (iii) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

putting b^2 for $a^2(1-e^2)$.

4. One circle lies completely within another. If a variable circle move so that it touches the inner of the fixed circles externally and the outer internally, prove that the locus of the centre of the variable circle is an ellipse, having for foci the centres of the fixed circles.

5. A variable circle, centre P, touches the fixed circles

$$x^2+y^2-20x=0 \quad \text{and} \quad x^2+y^2-28x+160=0.$$

Find the equation of the ellipse which is a locus of P, and the eccentricity of the ellipse.

6. $ABCD$ is a jointed frame-work consisting of four rods AB, AC, DB, DC, of which the members AC and BD cross at P. If $AB=CD$ and $AC=BD$, and if A and B are fixed, prove that the locus of P is an ellipse whose foci are A and B. If $AB=a$, $AC=b$, find the eccentricity of the ellipse.

7. If P is a point on the ellipse, eccentricity e, and foci S and S', prove that

$$\tan\tfrac{1}{2}PSS' \cdot \tan\tfrac{1}{2}PS'S=\frac{1-e}{1+e}.$$

8. A is a fixed point within a fixed circle, centre B. Through A is described any circle, centre C, of the same radius as the fixed circle, and intersecting it at Q and R. If AC and QR cut at P, prove that the locus of P is an ellipse of foci A and B.

129. The Hyperbola. Let a variable point P move so that its distance SP from a fixed point S bears a constant ratio e, greater than unity, to its distance PM from a fixed line ZX; then the locus of P is a hyperbola of *eccentricity* e, *focus* S and *directrix* ZX. The form of the curve is shown in Fig 114.

Let A and A' divide SX, the perpendicular from the focus to the directrix, internally and externally, so that

$$SA=e \,.\, AX \quad \text{and} \quad SA'=e \,.\, A'X;$$

let C be the middle point of AA', and let $L'SL$ perpendicular to AA' meet the curve in L' and L.

Then AA' is called the *transverse axis* of the hyperbola and is denoted by $2a$; C is called the *centre* and $L'L$ the *latus rectum* of the hyperbola.

If $B'CB$ be perpendicular to AA', and $B'C = CB = b$, where $b^2 = a^2(e^2 - 1)$, then $B'B$ is called the *conjugate axis* of the hyperbola.

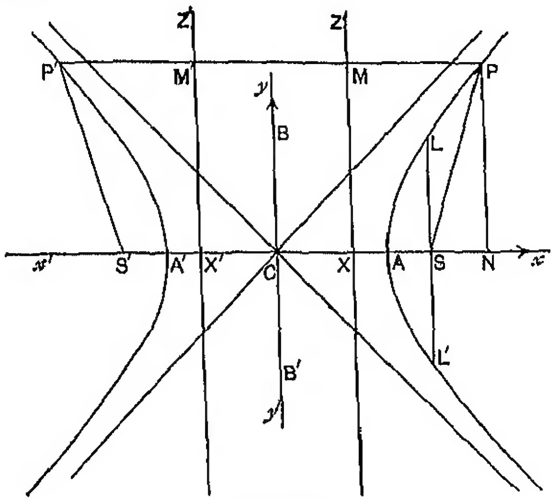


FIG. 114.

THEOREMS.

$$CS = e \,.\, CA = ea; \quad \text{.........................(1)}$$

$$CA = e \,.\, CX \text{ or } CX = a/e; \quad \text{...............(2)}$$

$$CS \,.\, CX = CA^2 = a^2; \quad \text{...............................(3)}$$

$$SL = a(e^2 - 1) = \frac{b^2}{a}. \quad \text{..........................(4)}$$

Proof. (1) $SA = e \,.\, AX$ and $SA' = e \,.\, A'X$;

therefore, by addition,

$$SA + SA' = e(AX + A'X), \text{ that is, } 2CS = 2e \,.\, CA,$$

whence
$$CS = e \,.\, CA = ea.$$

(2) Also, by subtraction,

$$SA' - SA = e(A'X - AX), \text{ that is, } 2CA = 2e \,.\, CX,$$

whence $$CA = e \,.\, CX \quad \text{or} \quad CX = \frac{a}{e}.$$

(3) $$CS \,.\, e \,.\, CX = e \,.\, CA \,.\, CA, \quad \text{by (1) and (2);}$$

hence $$CS \,.\, CX = CA^2 = a^2.$$

(4) $$SL = e \,.\, SX = e(CS - CX) = e\left(ae - \frac{a}{e}\right) = a(e^2 - 1) = \frac{b^2}{a}$$

since $$b^2 = a^2(e^2 - 1).$$

EXERCISES XXXVII.

1. If $CS=6$ and $CA=5$, calculate e and the distances of S and C from the directrix.

2. If $CA=6$ and $CS=7$, calculate e and the distances of S and A from the directrix.

3. If $a=4$ and $b=3$, calculate e and the distance from the focus to the directrix.

4. If $a=3$ and $b=4$, calculate e and the distance from the focus to the directrix.

5. If $CA=6$ and $e=2$, calculate CB, SX and SA.

6. If $CS=6$ and $CX=4$, calculate CA, CB and e.

7. If $a=5$ and $e=\frac{13}{5}$, calculate b, CS, CX, AX.

8. If $b=4$ and $e=\frac{5}{3}$, calculate a, CS, SA, SX.

9. If $a=3$ and $b=5$, calculate the length of the latus rectum.

10. If the focus of a hyperbola, whose eccentricity is 2, is 3 in. from the directrix, calculate the lengths of the transverse and conjugate axes.

11. Prove that $AS \,.\, A'S = CB^2$ and $AS/A'S = (e-1)/(e+1)$.

12. If $e > \sqrt{2}$, show that $b > a$.

13. Draw on squared paper the hyperbola where $CA=2$, $CS=3$.

14. If l is the semi-latus rectum of the hyperbola specified by $CA=a$, $CB=b$ and eccentricity $=e$, prove that

$$a = \frac{l}{e^2-1}; \quad b = \frac{l}{\sqrt{e^2-1}}; \quad CS = \frac{el}{e^2-1}.$$

In the following exercises CA, CB are taken as axes of x and y; CN, NP are the abscissa and ordinate of $P(x, y)$, a point on the hyperbola.

15. If $a=4$, $b=3$, $SP=5$, then $e=5/4$, $NX=4$, $CN=36/5$, $NP=\pm 6\sqrt{14}/5$ and $x^2/a^2-y^2/b^2=1$.

16. If $a=12$, $b=5$, $SP=13$, then $e=13/12$, $NX=12$, $CN=300/13$, $NP=\pm 10\sqrt{114}/13$ and $x^2/a^2-y^2/b^2=1$.

17. If (x, y) are the coordinates of L, prove that $x^2/a^2-y^2/b^2=1$.

18. If $b>a$ and $y=x$ meets the hyperbola in the point (x, y), prove that

$$x=\pm\frac{ab}{\sqrt{b^2-a^2}}=y \quad\text{and}\quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

130. Canonical Equation of the Hyperbola. If the hyperbola, whose semi-axes are $CA=a$ and $CB=b$, be referred to CA and CB as axes of x and y, its equation, then called the *canonical equation*, is

$$\frac{\mathbf{x}^2}{\mathbf{a}^2}-\frac{\mathbf{y}^2}{\mathbf{b}^2}=1.$$

Let $P(x, y)$ be any point on the curve (Fig. 114); let $CN=x$, $NP=y$. Then

$$\begin{aligned}
y^2=NP^2=SP^2-NS^2&=e^2\,.\,PM^2-NS^2\\
&=e^2\,.\,NX^2-NS^2\\
&=e^2(CN-CX)^2-(CN-CS)^2\\
&=e^2\left(x-\frac{a}{e}\right)^2-(x-ae)^2,\ \text{by (1) and (2) of §129,}\\
&=(ex-a)^2-(x-ae)^2\\
&=e^2x^2+a^2-x^2-a^2e^2\\
&=(x^2-a^2)(e^2-1)\\
&=(x^2-a^2)\cdot\frac{b^2}{a^2};
\end{aligned}$$

therefore $$\frac{y^2}{b^2}=\frac{x^2-a^2}{a^2}=\frac{x^2}{a^2}-1,$$

or $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Note. (1) The equation may be put in the form

$$\frac{y^2}{x^2-a^2}=\frac{b^2}{a^2} \quad \text{or} \quad \frac{NP^2}{CN^2-CA^2}=\frac{CB^2}{CA^2},$$

or, as it is usually written,

$$\frac{NP^2}{AN \cdot A'N}=\frac{CB^2}{CA^2}.$$

(2) The curve is symmetrical about both axes; for the image of P in AA', viz. $(x, -y)$, is on the curve, and the image of P in BB', viz $(-x, y)$, is also on the curve.

(3) Every straight line drawn through C and terminated by the curve is bisected at C, such a line being called a *diameter* of the hyperbola; for if PCP' is bisected at C, P' is the point $(-x, -y)$, and $(-x, -y)$ is on the curve.

(4) $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}<1$ for any point (x_1, y_1) outside both branches of the curve.

(5) $\frac{x_2^2}{a^2}-\frac{y_2^2}{b^2}>1$ for any point (x_2, y_2) inside either branch of the curve.

EXERCISES XXXVIII.

1. Find the eccentricity of the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$.

2. Find the eccentricity and the semi-latus rectum of the hyperbola

$$\frac{x^2}{16}-\frac{y^2}{9}=1.$$

3. Find the canonical equation of the hyperbola whose semi-conjugate axis is 3 and semi-latus rectum 1.

4. Find the distance from the focus to the directrix of the hyperbola $\frac{x^2}{144}-\frac{y^2}{25}=1$.

5. Find which of the following points are inside and which are not inside a branch of the hyperbola specified by $a=4$, $b=3$: $(2, -5)$, $(-5, 2)$, $(6, -4)$, $(-7, -4)$.

6. If $m<\frac{b}{a}$, find the coordinates of the points of intersection of $y=mx$ and $y=-mx$ with the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Discuss the case where $m=\frac{b}{a}$.

7. Show that the bisectors of the angles formed by CA and CB meet the curve in four real and distinct points if $b > a$.

8. If $a = b$, prove that $e = \sqrt{2}$.

9. Find the distance from the centre of the hyperbola specified by $a = 4$, $b = 3$, to the points on the curve whose abscissa is 6.

10. Establish the following formula for the length r of the semi-diameter of gradient $\tan\theta$:

$$r^2 = \frac{b^2}{e^2\cos^2\theta - 1}.$$

When is r real and finite, infinite, imaginary?

This is the *polar equation* of the hyperbola referred to C as pole and CA as initial line.

11. Establish the formulae:

$$SP = \sqrt{\{(ae - x)^2 + y^2\}};$$
$$SP = ex - a.$$

12. Find the equations and lengths of the axes, and also the co-ordinates of the centres of the following hyperbolas. Sketch the curves

(i) $\dfrac{(x-1)^2}{9} - \dfrac{(y+2)^2}{4} = 1$; (ii) $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$;

(iii) $3x^2 - 4y^2 - 6x - 8y = 13$; (iv) $8x^2 - 45y^2 - 8x - 60y = 28$;

(v) $\dfrac{(x-2y)^2}{40} - \dfrac{(y+2x)^2}{15} = 1$; (vi) $\dfrac{(x-y+1)^2}{16} - \dfrac{(x+y-2)^2}{9} = 1.$

13. Draw in one diagram the hyperbola $x^2/a^2 - y^2/b^2 = 1$, and the lines $y = \pm bx/a$.

14. Draw in one diagram the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$, and the lines $y = \pm bx/a$.

15. If a radius vector $CP = r$ be drawn to meet the hyperbola $x^2/a^2 - y^2/b^2 = 1$ in P, and another radius vector $CP' = r'$, perpendicular to CP, to meet the hyperbola $y^2/b^2 - x^2/a^2 = 1$ in P', prove that

$$\frac{1}{r^2} \sim \frac{1}{r'^2} = \frac{1}{a^2} \sim \frac{1}{b^2}.$$

16. Draw on squared paper the hyperbola whose focus is $(-2, 1)$, directrix $x + y = 2$, and eccentricity 2. Find the equation of the curve and its latus rectum.

17. Draw on squared paper the hyperbola whose focus is the origin, directrix $x + y = 3$, and eccentricity $\sqrt{2}$. Find its equation and the coordinates of its centre.

18. Prove that the equations

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

give the tangents of gradient m to the hyperbola $x^2/a^2 - y^2/b^2 = 1$.

19. Prove that the line $lx + my = n$ touches the hyperbola

$$x^2/a^2 - y^2/b^2 = 1 \quad \text{if} \quad a^2l^2 - b^2m^2 = n^2.$$

131. Foci of the Hyperbola. If the hyperbola (Fig. 114) be folded about the conjugate axis BB' so that A falls on A', one branch of the figure will be superposed on the other. Let P fall on P', S on S', ZX on $Z'X'$, PM on $P'M'$. Then $S'P' = e.P'M'$, so that the hyperbola may be traced from S' and $Z'X'$ as focus and directrix instead of from S and ZX. The hyperbola has thus two foci and two corresponding directrices, and the following theorem, which was taken in § 72 as the definition of a hyperbola, can now be proved.

THEOREM.

The difference of the focal distances of any point on a hyperbola is constant and equal to the transverse axis; or

$$SP \sim S'P = AA'.$$

Proof. We have (Fig. 114), if $CN = x$,

$$SP = e.MP = e.(CN - CX) = e.CN - e.CX = ex - a, \quad (1)$$

$$S'P = e.M'P = e.(CN + X'C) = e.CN + e.CX = ex + a, \quad (2)$$

because $e.CX = CA = a$. By subtraction, we now get

$$S'P - SP = 2a = AA'.$$

If P were on the other branch, the abscissa x would be negative, and we should have

$$SP = -ex + a, \ldots\ldots (1') \qquad S'P = -ex - a \ldots\ldots (2')$$

and

$$SP - S'P = 2a = AA'.$$

The expressions (1), (2), (1′), (2′) for the focal distances are of some importance; these expressions are all *positive*.

EXERCISES XXXIX.

1. In the mechanical description of a hyperbola, § 72, the ruler $S'K$ is 12 in. long, the distance between the fixed end S' of the ruler and the fixed end S of the string is 10 in. and the string is 4 in. long; find (i) the eccentricity, (ii) the transverse axis, (iii) the conjugate axis of the hyperbola described.

2. If the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ is mechanically described, what is the distance between the fixed ends of ruler and string?

3. If S and S' are the points $(ae, 0)$, $(-ae, 0)$, and a variable point $P(x, y)$ move so that $SP \sim S'P=2a$, establish directly the equations:

$$\text{(i) } SP=\sqrt{\{(ae-x)^2+y^2\}}; \qquad \text{(ii) } S'P=\sqrt{\{(ae+x)^2+y^2\}};$$
$$\text{(iii) } x^2/a^2-y^2/b^2=1, \text{ putting } b^2=a^2(e^2-1).$$

4. One circle lies completely outside another. If a variable circle move so that it touches both circles externally, or both internally, prove that the locus of the centre of the variable circle is a hyperbola, having for foci the centres of the fixed circles. Discuss the case where the contacts are one internal and one external.

5. A variable circle, centre P, touches the fixed circles

$$x^2+y^2-8x+12=0 \quad \text{and} \quad x^2+y^2+8x+15=0;$$

find the equations of the hyperbolas which form the locus of P, and their eccentricities.

6. Through a given point P, outside a fixed circle, centre C, is described any circle of the same radius as the fixed circle. If the line joining the centre of the variable circle to P meet the common chord of the two circles in Q, prove that the locus of Q is a hyperbola whose foci are C and P.

7. A parabola passes through two fixed points and has its axis in a given direction. Prove that the locus of its focus is a hyperbola.

8. A variable circle touches two fixed straight lines on which A and B are fixed points. The second tangents drawn from A, B to the circle meet in P. Prove that the locus of P is a hyperbola.

132. The Asymptotes of the Hyperbola. The equation $x^2/a^2-y^2/b^2=1$, may be written in the form

$$\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right)=1. \quad \text{..................(1)}$$

The two lines

$$\frac{x}{a}-\frac{y}{b}=0\text{......(2)} \quad \text{and} \quad \frac{x}{a}+\frac{y}{b}=0 \text{ ....(3)}$$

are the asymptotes of (1). For if we solve equations (1) and (2) or equations (1) and (3) as simultaneous equations, we get the anomalous equation $0=1$, where a quadratic is in question. Hence (§ 117) the lines (2) and (3) both meet (1) at two points at infinity, and may be considered tangents to (1) whose points of contact are at infinity. Lines (2) and (3) are shown in Fig. 114. The asymptotes have certain important properties which we proceed to investigate.

THEOREM 1.

If parallels to the asymptotes of a hyperbola drawn from a point P on the curve meet them at M and N, then $PM \, . \, PN$ is constant.

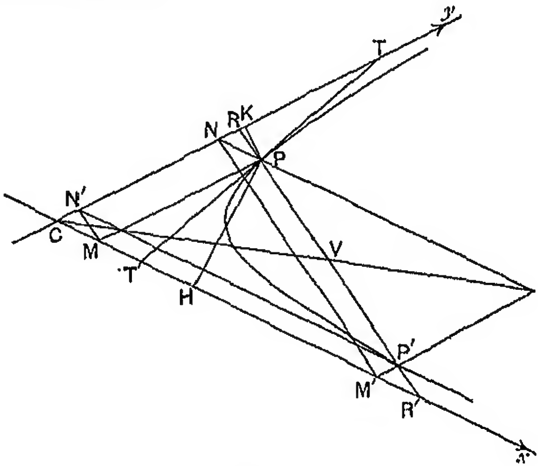


FIG. 115.

Let PH, PK (Fig. 115) be the perpendiculars from the point $P(x, y)$ on the hyperbola (1) to the asymptotes (2) and (3). Then (*numerically*)

$$PH \, . \, PK = \frac{\dfrac{x}{a}+\dfrac{y}{b}}{\sqrt{\left\{\dfrac{1}{a^2}+\dfrac{1}{b^2}\right\}}} \cdot \frac{\dfrac{x}{a}-\dfrac{y}{b}}{\sqrt{\left\{\dfrac{1}{a^2}+\dfrac{1}{b^2}\right\}}} = \frac{\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}}{\dfrac{1}{a^2}+\dfrac{1}{b^2}} = \frac{a^2b^2}{a^2+b^2},$$

because $x^2/a^2 - y^2/b^2 = 1$.

Now let 2ω be the angle between the asymptotes; then (*numerically*)

$$PM \sin 2\omega = PH \quad \text{and} \quad PN \sin 2\omega = PK;$$

therefore $\quad PM \,.\, PN \sin^2 2\omega = PH \,.\, PK = \dfrac{a^2b^2}{a^2+b^2},$

so that $\quad PM \,.\, PN$ is constant.

Since $\tan \omega = b/a$,

$$\sin 2\omega = \frac{2 \tan \omega}{1+\tan^2 \omega} = \frac{2ab}{a^2+b^2}.$$

Hence, $\quad PM \,.\, PN = \dfrac{a^2+b^2}{4},$

so that the value of the constant is $(a^2+b^2)/4$.

The theorem may be stated analytically as

THEOREM 2.

If a hyperbola, of semi-axes a and b, be referred to its asymptotes as axes of x and y, its constraint equation may be written in the form,

$$\mathbf{xy} = \mathbf{c}^2,$$

and its freedom equations in the form

$$\mathbf{x} = \mathbf{ct}, \quad \mathbf{y} = \frac{\mathbf{c}}{\mathbf{t}},$$

where $c^2 = (a^2+b^2)/4$.

These axes are rectangular if, and only if, the asymptotes are at right angles to each other; in this case the hyperbola is called a **rectangular hyperbola**, or an **equilateral hyperbola**.

If the axes are not the asymptotes but lines parallel to the asymptotes, then, as is easily seen by changing the origin, the equation of the hyperbola will take the form

$$\mathbf{axy} + \mathbf{bx} + \mathbf{cy} + \mathbf{d} = 0,$$

an equation which may also be written in the form

$$\mathbf{y} = \frac{\mathbf{px}+\mathbf{q}}{\mathbf{rx}+\mathbf{s}}.$$

THEOREM 3.

If PM, PN, $P'M'$, $P'N'$ are drawn parallel to the asymptotes from points P, P' on the curve, then MN', $M'N$ and PP' are parallel.

For, in Fig. 115, MN', $M'N$ are parallel if

$$CM : CM' = CN' : CN,$$

that is, if $CM . CN = CM' . CN'$; and this is true, since P, P' lie on the curve $xy = c^2$.

Again, parallelogram $CMPN$ = parallelogram $CM'P'N'$, since $CM . CN = CM' . CN'$.

And the parallelograms $CMPN$, $CM'P'N'$ are double the triangles $NM'P$, $NM'P'$ respectively; so that the triangles $NM'P$, $NM'P'$ are equal, and therefore $M'N$ is parallel to PP'.

THEOREM 4.

If the chord PP' of a hyperbola meet the asymptotes in R, R', then $RP = R'P'$.

Using the notation of Theorem 3, we see that MN', $M'N$, PP' in Fig. 115 are the diagonals of parallelograms, and that the other diagonals form one and the same straight line through C. Hence the same diameter of the hyperbola bisects PP' and MN'. But the diameter which bisects MN' also bisects the parallel RR'; therefore PP' and RR' have the same middle point, V, so that $RP = R'P'$.

THEOREM 5.

The locus of the middle points of parallel chords is a diameter.

The diameter which bisects PP' in Fig. 115 also bisects MN'. As PP' moves parallel to itself, so does MN', and MN' is constantly bisected by the same diameter; therefore so is PP'

THEOREM 6.

The portion of a tangent intercepted between the asymptotes is bisected at the point of contact, and the

tangent forms with the asymptotes a triangle of constant area.

The first part follows from each of Theorems 3 and 4. Take Theorem 3 and let P' (Fig. 115) move into coincidence with P; then M' coincides with M and N' with N; PP' becomes the tangent at P, so that the tangent at P is parallel to MN. If the tangent meet the asymptotes at T, T', then PT and PT' are both equal to MN.

Again, triangle CTT' is twice the parallelogram $CMPN$, and therefore its area is constant.

133. Polar Equation of a Conic. The polar equation of a conic with the focus as pole is often useful, and though we shall make little use of it in this book, we shall establish it here so that it may be referred to when required.

Take S' in Fig. 113 as pole and $S'X'$ as initial line; denote $S'P$ by r and angle $X'S'P$ by θ. Then

$$S'P = e \,.\, M'P = e \,.\, X'N = e(X'S' + S'N) = e \,.\, X'S' + e \,.\, S'N. \quad (1)$$

But $e \,.\, X'S'$ is equal to the semi-latus rectum, which we shall denote by l; also

$$S'N = S'P \cos NS'P = r \cos(\pi - \theta) = -r \cos\theta. \quad \ldots\ldots(2)$$

Equation (1) now becomes

$$r = l - er\cos\theta \quad \text{or} \quad r(1 + e\cos\theta) = l, \quad \ldots\ldots\ldots\ldots(3)$$

so that the required equation is

$$r = \frac{l}{1 + e\cos\theta}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The proof holds for any conic; for the parabola $e = 1$.

If the initial line is not $S'X'$, but a line which makes with $S'X'$ the angle α, then the vectorial angle θ will not be $X'S'P$ but $X'S'P$ increased or diminished by α; instead of θ in equation (4) we shall have $(\theta \pm \alpha)$.

The changes in equation (4) when the vectorial angle is not $X'S'P$ but $XS'P$, or when the focus S, instead of S' is pole, will be easily made.

Ex. Show that the semi-latus rectum is the harmonic mean between the two segments $S'P$, $S'Q$ of any focal chord PQ.

Let θ be the angle $X'S'P$; then $(\theta+\pi)$ is the angle $X'S'Q$. From equation (4) we have

$$\frac{1}{S'P}=\frac{1+e\cos\theta}{l},\quad \frac{1}{S'Q}=\frac{1+e\cos(\theta+\pi)}{l},$$

and therefore
$$\frac{1}{S'P}+\frac{1}{S'Q}=\frac{2}{l},$$

since $\cos(\theta+\pi)=-\cos\theta$. Hence (§ 44) l is the harmonic mean between $S'P$ and $S'Q$.

EXERCISES XL.

1. Prove that the line $y=mx$ meets the curve $x^2/a^2-y^2/b^2=1$ in real points if m lies between b/a and $-b/a$.

2. What are the asymptotes of the hyperbolas:

(i) $4x^2-9y^2=36$; (ii) $x^2-y^2=1$; (iii) $\frac{(x-1)^2}{4}-\frac{(y+1)^2}{9}=1$;

(iv) $\frac{(2x-y)^2}{16}-\frac{(x+2y)^2}{9}=1$; (v) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$?

Draw the curves and the asymptotes.

3. Prove that the conjugate hyperbola $x^2/a^2-y^2/b^2=-1$ has the same asymptotes as $x^2/a^2-y^2/b^2=+1$. Show the asymptotes and both curves in one diagram.

4. If the asymptotes of a hyperbola of eccentricity e meet at an angle 2ω, prove that $\sin\omega=\sqrt{e^2-1}/e$, $\sec\omega=e$, and $\tan\omega=\sqrt{e^2-1}$.

5. If the two hyperbolas

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad\text{and}\quad \frac{x^2}{a_1^2}-\frac{y^2}{b_1^2}=1$$

have the same eccentricity, prove that $a/a_1=b/b_1$. If a radius vector meet the first in P and the second in Q, prove that $CP:CQ$ is constant. (The hyperbolas are said to be *homothetic* or *similar and similarly situated*.)

6. Discuss the variation of the form of the hyperbola $x^2/a^2-y^2/b^2=k$, as k diminishes from 1 to 0.

7. A rectangle is formed by drawing parallels to AA' through B, B' and to BB' through A, A'; prove that the diagonals of the rectangle are the asymptotes.

8. If the axes of x and y are at right angles, show that the length of each axis of the hyperbola $xy=c^2$ is $2c\sqrt{2}$.

9. Prove that $xy+2x-3y=0$ represents a rectangular hyperbola. Find the equations of its asymptotes and the coordinates of its centre. Draw the curve and its asymptotes.

10. Find the asymptotes of the hyperbola

$$axy+bx+cy+d=0.$$

11. Find the lengths of the semi-axes of the rectangular hyperbolas

(i) $xy=1$; (ii) $6xy+4x-9y=7$.

12. Prove that the product of the perpendiculars from any point of the curve $3x^2+4xy=6$ on to the asymptotes of the curve is the constant $6/5$. Find the squares of the semi-axes of the hyperbola represented by $3x^2+4xy=6$.

13. Find the asymptotes and centres of the hyperbolas:

(i) $(2x-y+1)(x+y-2)=3$; (ii) $3x^2-4xy-4y^2-5x-6y=3$.

14. Find the equation of the hyperbola whose asymptotes are parallel to $2x+3y=0$, $x-2y=0$, whose centre is at $(1, 2)$, and which passes through $(5, 3)$.

15. Show that the two chords of the hyperbola

$$xy-2x-3y+5=0$$

which pass through $(0, 2)$ and subtend a right angle at the origin are inclined to the x-axis at angles $135°$ and $\cot^{-1} 5$.

16. P is any point on the fixed line $y=mx$, A and B are the fixed points $(c, 0)$ and $(-c, 0)$. The line PQ subtends a right angle at each of the points A and B; prove that the locus of Q is a hyperbola one of whose asymptotes is the y-axis and the other the perpendicular through the origin to the locus of P. Sketch the locus of Q.

17. Prove that the equation of the tangent to

$$xy=c^2,$$

at the point (x_1, y_1), is $x/x_1+y/y_1=2$.

18. Prove that the equation of the chord of $xy=c^2$, whose extremities are $(ct_1, c/t_1)$ and $(ct_2, c/t_2)$, is

$$y=c\left(\frac{1}{t_1}+\frac{1}{t_2}\right)-\frac{x}{t_1t_2}.$$

Deduce the equation of the tangent at the point t.

19. A parallelogram has its sides parallel to the asymptotes of a hyperbola and the extremities of one diagonal lie on the curve: prove that the other diagonal passes through the centre. Deduce that the locus of middle points of parallel chords of a hyperbola is a diameter.

20. A chord PQ of a hyperbola meets an asymptote in R, and M, N are points on this asymptote such that MP, NQ are parallel to the other asymptote; prove that $OM=NR$.

21. Any two points P, Q are taken on a hyperbola, centre C. Lines through P, Q parallel to the asymptotes meet in K. Prove that CK bisects PQ.

22. E and F are fixed points on a hyperbola, P a variable point. PE, PF meet an asymptote in M, N. Prove that MN is of constant length. By considering P at infinity on the curve, verify what the constant length is.

23. Show that the tangents at the extremities of a chord of a hyperbola meet either asymptote in points equally distant from its intersection with the chord.

24. MM' is any chord of a hyperbola and P is an extremity of the diameter bisecting the chord. Lines MK, PQ, $M'K'$ are drawn parallel to one asymptote to meet the other in K, Q, K'. Show that $CK \cdot CK' = CQ^2$, where C is the centre of the hyperbola.

25. Prove that the tangents at the vertices of a hyperbola meet its asymptotes on the circumference of the circle of which the line joining the foci is a diameter.

26. Given one asymptote of a hyperbola, two tangents and the point of contact of one of them, construct the other asymptote.

27. The tangent at a point P on a hyperbola meets the asymptotes in T, T' and the circum-circle of the triangle CTT' cuts the axes at G and g; prove that the line Gg is the normal at the point P (that is, the perpendicular at P to the tangent).

28. The straight line $x/a + y/b = 1$ meets the axes at A, B, and C is the middle point of AB. Find the equation of the hyperbola which passes through C and has the axes as asymptotes. If the axes are rectangular, find the length of either semi-axis.

29. If r, r' be focal radii of an ellipse at right angles to each other, prove that

$$\left(\frac{1}{r}-\frac{1}{l}\right)^2+\left(\frac{1}{r'}-\frac{1}{l}\right)^2$$

is constant, where l is the semi-latus rectum.

30. If f, f' be two focal chords of a parabola at right angles to each other, prove that

$$\frac{1}{f}+\frac{1}{f'}=\frac{1}{2l},$$

where l is the semi-latus rectum.

CHAPTER XIX.

THE ELLIPSE AS THE ORTHOGONAL PROJECTION OF A CIRCLE.

134. The Auxiliary Circle. We shall now consider the ellipse from another point of view. On the major axis AA' of the ellipse $x^2/a^2+y^2/b^2=1$, as diameter, describe a circle; its equation is

$$x^2+y^2=a^2.$$

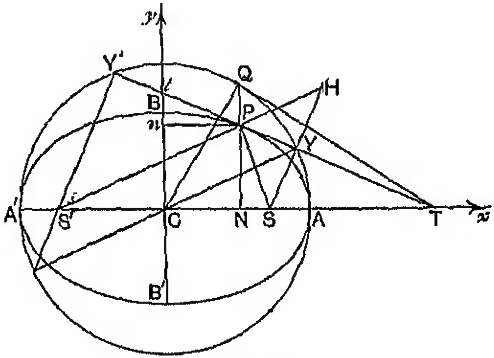


FIG. 116.

Let NP, an ordinate of the ellipse, meet this circle in Q (Fig. 116). Then

$$\frac{CN^2}{a^2}+\frac{NP^2}{b^2}=1 \quad \text{and} \quad \frac{CN^2}{a^2}+\frac{NQ^2}{a^2}=1;$$

therefore $$\frac{NP^2}{b^2}=\frac{NQ^2}{a^2} \quad \text{or} \quad NP=\frac{b}{a}\cdot NQ.$$

Hence the ellipse $x^2/a^2+y^2/b^2=1$ may be derived from the circle $x^2+y^2=a^2$ by shortening each ordinate NQ of the circle in the ratio $b:a$. Or we may suppose the circle AQA' to be tilted round AA', out of the horizontal plane of the paper, till Q lies vertically over P, then the upper half AQA' of the circle will lie vertically over APA', the upper half of the ellipse, and the under half of the circle will lie vertically below the under half of the ellipse. If vertical rays descend upon the circle in its tilted position, those which graze the circumference of the circle will mark the outline of the ellipse. Hence an ellipse is said to be the *orthogonal projection* of a circle. The circle on AA' as diameter is called the **auxiliary circle** and P, Q are called **corresponding points.**

The angle ACQ is called the **eccentric angle** of P, and if this angle is denoted by θ the coordinates of P are

$$\mathbf{x}=\mathbf{a}\cos\theta,\quad \mathbf{y}=\mathbf{b}\sin\theta.$$

For $$\cos\theta=\frac{CN}{CQ}=\frac{x}{a}, \text{ and therefore } x=a\cos\theta,$$

$$\sin\theta=\frac{NQ}{CQ}=\frac{NQ}{a}, \text{ and therefore } NQ=a\sin\theta;$$

but $$y=NP=\frac{b}{a}\cdot NQ=b\sin\theta.$$

P is often called "the point θ."

The equations $x=a\cos\theta,\quad y=b\sin\theta$

are **Freedom Equations** of the ellipse.

Ex. 1. Show that the area of the ellipse is πab.

Compare the areas of two strips, one bounded by the x-axis, the auxiliary circle and two ordinates NQ, $N'Q'$, and the other bounded by the x-axis, the ellipse and the two corresponding ordinates NP, $N'P'$. The line NN' is the same for both strips, and all corresponding ordinates are in the ratio $b:a$, so that

$$\text{area of ellipse : area of circle} = b : a.$$

Therefore $$\text{area of ellipse}=\frac{b}{a}\cdot(\text{area of circle})=\frac{b}{a}\cdot\pi a^2=\pi ab.$$

Ex. 2. If M is the projection of P on the minor axis BB', and if MP meet the circle on BB' as diameter in P', show that $MP:MP'$ is constant and equal to $a:b$. Deduce the theorem that the circle on BB' as diameter is the orthogonal projection of the ellipse.

We have, from the equations of ellipse and circle,

$$\frac{MP^2}{a^2}+\frac{CM^2}{b^2}=1, \quad \frac{MP'^2}{b^2}+\frac{CM^2}{b^2}=1,$$

so that
$$\frac{MP^2}{a^2}=\frac{MP'^2}{b^2}, \quad \frac{MP}{MP'}=\frac{a}{b}.$$

By tilting the ellipse round BB' till P lies vertically over P', we prove the projection theorem, as we proved the corresponding theorem in the text on the relation of the auxiliary circle to the ellipse.

135. The Tangent and Normal. The normal at any point P on a curve is the perpendicular through P to the tangent to the curve at P. We now prove

THEOREM 1.

Tangents at P and Q, corresponding points on the ellipse and the auxiliary circle, meet on the major axis at a point T such that

$$CN \,.\, CT = CA^2,$$

where N is the projection of P or Q on the major axis (Fig. 116).

Let QQ', a secant of the circle, meet AA' in T'. Now tilt the circle, and the secant with it, round $T'AA'$ till the circle projects into the ellipse; $T'QQ'$ then projects into $T'PP'$, the corresponding secant of the ellipse. Let T' move along the major axis so that $T'QQ'$ turns round Q, bringing Q' finally into coincidence with Q; T' will reach some point, T say, on the axis AA'. Since P, Q and P', Q' are pairs of corresponding points, P' will come into coincidence with P when Q' comes into coincidence with Q. Hence the tangents at P and Q to the ellipse and the circle will meet at T.

From the similar triangles CNQ, CQT, we now have

$$CN : CQ = CQ : CT$$

or
$$CN \,.\, CT = CQ^2 = CA^2.$$

In the same way, by making use of the circle on BB' as diameter (§ 134, Ex. 2), we deduce

THEOREM 2.

The tangents at P and Q, corresponding points on the ellipse and the circle on BB' as diameter, meet the minor axis at a point t such that

$$Cn \,.\, Ct = CB^2,$$

where n is the projection of P or Q on the minor axis.

From Theorems 1 and 2 we readily deduce the equations of the tangent and normal at any point $P(a\cos\theta,\ b\sin\theta)$ on the ellipse $x^2/a^2+y^2/b^2=1$.

Let the tangent (Fig. 116) meet the major and minor axes in T and t respectively;

then $$CT=\frac{CA^2}{CN}=\frac{a}{\cos\theta},\quad Ct=\frac{CB^2}{Cn}=\frac{b}{\sin\theta},$$

and therefore the equation of the tangent is (§ 32, (4))

$$\frac{x}{CT}+\frac{y}{Ct}=1 \quad\text{.....................(1)}$$

or $$\frac{\mathbf{x}\cos\theta}{\mathbf{a}}+\frac{\mathbf{y}\sin\theta}{\mathbf{b}}=\mathbf{1}. \quad\text{.....................(2)}$$

If P is the point (x_1, y_1), then $CT=a^2/x_1$, $Ct=b^2/y_1$, and equation (1) becomes

$$\frac{\mathbf{x_1x}}{\mathbf{a^2}}+\frac{\mathbf{y_1y}}{\mathbf{b^2}}=\mathbf{1}. \quad\text{..........................(3)}$$

The normal at P is the perpendicular to the line (2) through the point $(a\cos\theta,\ b\sin\theta)$; **the equation of the normal is therefore**

$$(x-a\cos\theta)\frac{a}{\cos\theta}-(y-b\sin\theta)\frac{b}{\sin\theta}=0$$

or $$\frac{\mathbf{ax}}{\mathbf{\cos\theta}}-\frac{\mathbf{by}}{\mathbf{\sin\theta}}=\mathbf{a^2-b^2}. \quad\text{.....................(4)}$$

The equation of the normal at (x_1, y_1) is

$$\frac{\mathbf{a^2x}}{\mathbf{x_1}}-\frac{\mathbf{b^2y}}{\mathbf{y_1}}=\mathbf{a^2-b^2}. \quad\text{.....................(5)}$$

Ex. 1. Show that the condition that the line

$$lx+my=n \qquad\text{(i)}$$

should be a tangent to the ellipse $x^2/a^2+y^2/b^2=1$ is

$$l^2a^2+m^2b^2=n^2.$$

Equation (i) must, if the line touch the ellipse at the point θ, be equivalent to equation (2) above. Hence

$$\frac{l}{n}=\frac{\cos\theta}{a},\quad \frac{m}{n}=\frac{\sin\theta}{b}, \qquad\text{(ii)}$$

and therefore $\qquad a^2l^2+b^2m^2=n^2(\cos^2\theta+\sin^2\theta)=n^2.$

The point of contact is given by equations (ii).

Otherwise. Express the condition that the equations of the line and the ellipse, regarded as simultaneous equations, should have a repeated solution. The equation for x is

$$(l^2a^2+m^2b^2)x^2-2nla^2x+a^2(n^2-m^2b^2)=0,$$

and the condition for equal roots is

$$n^2l^2a^4=a^2(l^2a^2+m^2b^2)(n^2-m^2b^2)$$

or

$$l^2a^2+m^2b^2=n^2.$$

Ex. 2. If SY, $S'Y'$ are the perpendiculars from the foci on the tangent PT at P, show that

$$SY.S'Y'=CB^2.$$

S is the point $(ea, 0)$ and S' the point $(-ea, 0)$ (Fig 116); form the expressions (§ 30) for the lengths of the perpendiculars on the line given by equation (2) of the text, which may be written in the form

$$bx\cos\theta+ay\sin\theta-ab=0,$$

and we get

$$SY.S'Y'=\frac{eab\cos\theta-ab}{\sqrt{(b^2\cos^2\theta+a^2\sin^2\theta)}}\cdot\frac{-eab\cos\theta-ab}{\sqrt{(b^2\cos^2\theta+a^2\sin^2\theta)}}$$

$$=\frac{a^2b^2(1-e^2\cos^2\theta)}{b^2\cos^2\theta+a^2\sin^2\theta}.$$

But $e^2a^2=a^2-b^2$, and the numerator becomes

$$b^2(a^2-a^2\cos^2\theta+b^2\cos^2\theta)=b^2(a^2\sin^2\theta+b^2\cos^2\theta),$$

and therefore $SY.S'Y'=b^2$.

Ex. 3. If p is the perpendicular from the centre C on the tangent PT at P, show that $\quad p=ab/\sqrt{(a^2\sin^2\theta+b^2\cos^2\theta)}.$

C is the point $(0, 0)$; the required perpendicular p is therefore the numerical value of $\quad -ab/\sqrt{(b^2\cos^2\theta+a^2\sin^2\theta)}.$

Ex. 4. R is the point of intersection of the normals at the points P and Q on the ellipse $x^2/a^2+y^2/b^2=1$, and the line joining the centre

C to R bisects PQ; show that the tangents at P and Q intersect at right angles.

Let θ, ϕ be the eccentric angles of the points P, Q.

Then the equations of PR, QR, the normals at P and Q, are

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2 \quad\text{and}\quad \frac{ax}{\cos\phi}-\frac{by}{\sin\phi}=a^2-b^2.$$

Hence, by § 38, the equation of CR is

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=\frac{ax}{\cos\phi}-\frac{by}{\sin\phi}$$

or
$$\frac{ax(\cos\phi-\cos\theta)}{\cos\theta\cos\phi}=\frac{by(\sin\phi-\sin\theta)}{\sin\theta\sin\phi}.$$

But CR passes through the point $\left\{\frac{a}{2}(\cos\theta+\cos\phi), \frac{b}{2}(\sin\theta+\sin\phi)\right\}$, since this is the middle point of PQ.

Therefore
$$\frac{a^2(\cos^2\phi-\cos^2\theta)}{\cos\theta\cos\phi}=\frac{b^2(\sin^2\phi-\sin^2\theta)}{\sin\theta\sin\phi},$$

which reduces to
$$\frac{b\cos\theta}{a\sin\theta}\cdot\frac{b\cos\phi}{a\sin\phi}=-1.$$

Now the equation of the tangent at P is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1,$$

so that the gradient of the tangent is $-b\cos\theta/a\sin\theta$; and similarly the gradient of the tangent at Q is $-b\cos\phi/a\sin\phi$.

But we have shown that the product of these is -1.

Hence the theorem is proved.

EXERCISES XLI.

1. If $x=a\cos\theta$, $y=b\sin\theta$, prove that $x^2/a^2+y^2/b^2=1$.

2. Find the eccentric angle of the point $(4, -1{\cdot}2)$ on the ellipse $x^2/25+y^2/4=1$.

3. Prove that the eccentric angles of the extremities of the latera recta of the ellipse $x^2/a^2+y^2/b^2=1$ are given by the equation

$$\tan\theta=\pm b/ae.$$

4. Find the abscissae and ordinates of the points on the ellipse $x^2/a^2+y^2/b^2=1$ whose eccentric angles are 30°, 45°, 60°.

5. Prove that the eccentric angles of the extremities of a diameter differ by π.

6. Prove that, with the usual notation,

$$SP=a(1-e\cos\theta) \quad\text{and}\quad S'P=a(1+e\cos\theta).$$

7. If M is the point on the auxiliary circle corresponding to L, an extremity of the latus rectum, prove that $SM=CB$.

8. If P, Q, points on an ellipse, whose eccentric angles are θ, ϕ, correspond to the points p, q on the auxiliary circle, prove that the equations of PQ and pq are

$$\frac{x}{a}\cos\tfrac{1}{2}(\theta+\phi)+\frac{y}{b}\sin\tfrac{1}{2}(\theta+\phi)=\cos\tfrac{1}{2}(\theta-\phi),$$

$$\frac{x}{a}\cos\tfrac{1}{2}(\theta+\phi)+\frac{y}{a}\sin\tfrac{1}{2}(\theta+\phi)=\cos\tfrac{1}{2}(\theta-\phi).$$

Deduce (i) that PQ, pq intersect on the major axis, (ii) the equation of the tangent at the point θ.

9. If P and Q are corresponding points on the ellipse $x^2/a^2+y^2/b^2=1$ and its auxiliary circle, and m is the gradient of CP, find the gradient of CQ.

10. If P, Q are corresponding points on the ellipse $x^2/a^2+y^2/b^2=1$ and its auxiliary circle, and m is the gradient of the tangent at P to the ellipse, find the gradient of the tangent at Q to the circle.

11. If x_1 is the abscissa and θ the eccentric angle of P, prove that, with the usual notation,

$$CT=a^2/x_1=a/\cos\theta, \quad \text{and} \quad NT=(a^2-x_1^2)/x_1=a\sin^2\theta/\cos\theta.$$

NT is called the **subtangent**.

12. If θ is the eccentric angle of P, prove that the gradient of the tangent PT is $-b\cos\theta/a\sin\theta$.

13. Prove that the equation of the tangent at the point (x_1, y_1) on the ellipse $x^2/a^2+y^2/b^2=1$ is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1,$$

by using the equation of the tangent at the corresponding point on the auxiliary circle.

14. If CD is the semi-diameter parallel to PT, the tangent at the point θ, prove that D is the point $\theta+\frac{\pi}{2}$ or $(-a\sin\theta, b\cos\theta)$, and that

$$CP^2+CD^2=a^2+b^2.$$

15. Prove that the circle on the subtangent NT as diameter cuts the auxiliary circle orthogonally.

16. The tangent at P meets the major axis in T; AP and $A'P$ meet the perpendicular to AA' through T in Q and Q' respectively; prove that QQ' is bisected at T.

17. If F is the projection of the centre C on a variable tangent at P to the ellipse $x^2/a^2+y^2/b^2=1$, prove that the maximum value of PF is $a-b$.

18. Find the condition that $lx+my=1$ may be a tangent to the ellipse $x^2/a^2+y^2/b^2=1$.

19. A variable circle of the same radius as the circle

$$x^2+y^2+8x=84$$

passes through the point (4, 0); prove that the common chord of the circles touches the ellipse $x^2/25+y^2/9=1$.

20. Prove that the equations of the tangents to the ellipse

$$x^2/a^2+y^2/b^2=1$$

of gradient m are $$y=mx\pm\sqrt{a^2m^2+b^2}.$$

21. If AA', BB' are the principal axes of an ellipse and $AA'^2=2BB'^2$, show that the sum of the squares of the perpendiculars from B, B' on any tangent to the curve is constant.

22. Find the eccentric angle of Q if the tangent at Q is perpendicular to the tangent at P, whose eccentric angle is θ, and prove that the locus of intersection of the tangents at P, Q is the circle, called the **director-circle**, whose equation is

$$x^2+y^2=a^2+b^2.$$

23. Prove that the product of the distances of a chord of an ellipse from the two tangents parallel to it is the difference between the square of the semi-axis minor and the product of the perpendiculars on the chord from the foci.

24. The tangent and normal at a point P on an ellipse meet the minor axis in t and g; prove that tg subtends a right angle at each of the foci.

25. Any tangent to the ellipse $x^2/a^3+y^2/b^3=1/(a+b)$ meets the ellipse $x^2/a^2+y^2/b^2=1$ in two points, the normals at which are equidistant from the centre.

26. Find the condition that $lx+my=n$ may be a normal to the ellipse $x^2/a^2+y^2/b^2=1$.

27. If from any point on an ellipse perpendiculars are drawn to the axes, show that the line joining the feet of these perpendiculars is always normal to a fixed concentric ellipse.

28. Show that, if the normal to an ellipse meets the minor axis in g and S is a focus, $Sg=e\,.\,Pg$.

29. If PP' be a diameter of the ellipse $x^2/a^2+y^2/b^2=1$, prove that the locus of the intersection of the normal at P with the ordinate at P' is

$$\frac{x^2}{a^2}+\frac{b^2y^2}{(2a^2-b^2)^2}=1.$$

30. P is the point on the ellipse $x^2/a^2+y^2/b^2=1$ whose coordinates are $a^{\frac{3}{2}}/\sqrt{a+b}$, $b^{\frac{3}{2}}/\sqrt{a+b}$; find (i) the length of the perpendicular CQ

let fall from the centre C on the tangent at P, (ii) the length PQ, (iii) the equation of the normal at P, (iv) the coordinates of the point where the normal meets the curve again.

31. If SY, $S'Y'$ are the perpendiculars from the foci of an ellipse on the tangent at P, and if the normal at P meets the major axis in G, prove that

$$\frac{1}{SY^2}+\frac{1}{S'Y'^2}-\frac{4}{PG^2}$$

is constant.

32. If the chords joining the pairs of points α, β and γ, δ respectively meet the major axis in two points equidistant from the centre, prove that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2}=1.$$

33. Find the equation of the ellipse, whose semi-axes are a, b, referred to $A'A$ and the tangent at A' as axes of x and y. By considering the limiting form of the equation as e tends towards 1, while S', A' remain fixed, show that the parabola is a limiting form of the ellipse.

34. If θ, ϕ are the eccentric angles of the extremities of a chord through the focus, prove that

$$\cos\frac{\theta-\phi}{2}=e\,.\cos\frac{\theta+\phi}{2}.$$

35. PSQ, $PS'R$ are two focal chords of an ellipse and the eccentric angles of the points Q and R are ϕ_1 and ϕ_2. Show that $\tan\frac{\phi_1}{2}:\tan\frac{\phi_2}{2}$ is a constant ratio for all positions of P.

36. If PQ, PR are focal chords of an ellipse and 2α, 2β, 2γ are the eccentric angles of P, Q, R, prove that

$$\tan^2\alpha\tan\beta\tan\gamma=1.$$

37. If P, Q are the points θ, ϕ on the ellipse $x^2/a^2+y^2/b^2=1$ such that SP, $S'Q$ are parallel and in the same direction, prove (1) that $e=\sin\frac{1}{2}(\phi-\theta)/\sin\frac{1}{2}(\phi+\theta)$, (2) that PQ touches the ellipse

$$x^2/a^4+y^2/b^4=1/a^2,$$

and (3) that the tangents at P, Q intersect on the auxiliary circle.

38. P, P' are corresponding points on the ellipse $x^2/a^2+y^2/b^2=1$ and its auxiliary circle. If CP' meets the normal at P in Q, find the equation of the locus of Q.

136. Conjugate Diameters. Consider a diameter $Q'CQ$ of the auxiliary circle. It bisects all chords of the circle perpendicular to it, and these form a system of parallel chords of the circle. Now turn the circle, and the system

of parallel chords along with it, about AA' till the circle projects into the ellipse; the system of chords of the circle will project into a corresponding system of parallel chords of the ellipse, and the tangents at Q and Q', which are parallel to the chords of the circle, will project into

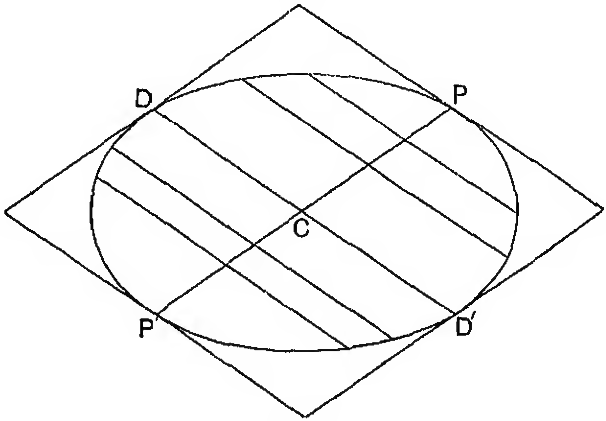


Fig. 117.

the tangents at P and P' on the ellipse, which will be parallel to the chords of the ellipse. Since the middle points of the chords of the circle lie on the straight line $Q'CQ$, the middle points of the chords of the ellipse will lie on the straight line $P'CP$. Hence we have (Fig. 117)

Theorem 3.

The locus of the middle points of a system of parallel chords of an ellipse is a diameter of the ellipse, and the tangents to the ellipse at the ends of the diameter are parallel to the chords bisected by the diameter.

Again, suppose $Q'CQ$ and $R'CR$ in Fig. 118 to be two *perpendicular* diameters of the auxiliary circle. Each of these bisects chords parallel to the other; therefore they project into diameters $P'CP$ and $D'CD$ of the ellipse, each of which bisects chords of the ellipse parallel to the other. Such diameters are called **conjugate diameters**, and we have

THEOREM 4.

If chords of an ellipse parallel to a diameter $D'CD$ are bisected by the diameter $P'CP$, then chords parallel to the diameter $P'CP$ are bisected by the diameter $D'CD$.

If the angle $ACQ = \theta$, then the angle $ACR = \theta + \pi/2$ and the angle $ACR' = \theta - \pi/2$; therefore if P is the point θ, D will be the point $\theta + \pi/2$ and D' the point $\theta - \pi/2$. We now prove

THEOREM 5.

If CP and CD are conjugate semi-diameters,

$$CP^2 + CD^2 = CA^2 + CB^2.$$

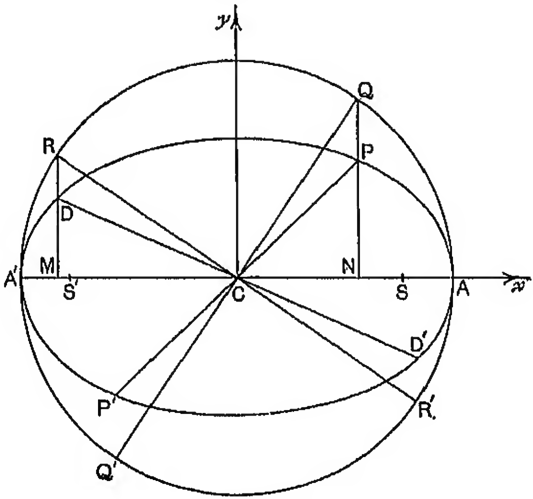


FIG. 118.

P is the point $(a\cos\theta,\ b\sin\theta)$ in Fig. 118 and D the point $(-a\sin\theta,\ b\cos\theta)$, because $\cos(\theta+\pi/2) = -\sin\theta$ and $\sin(\theta+\pi/2) = \cos\theta$. Therefore

$$CP^2 = a^2\cos^2\theta + b^2\sin^2\theta, \quad CD^2 = a^2\sin^2\theta + b^2\cos^2\theta$$

and $CP^2 + CD^2 = a^2(\cos^2\theta + \sin^2\theta) + b^2(\sin^2\theta + \cos^2\theta) = a^2 + b^2.$

THEOREM 6.

If CP, CD are conjugate semi-diameters and p is the perpendicular from the centre C on the tangent at P, then

$$p \, . \, CD = ab.$$

Since $CD = \sqrt{(a^2 \sin^2\theta + b^2 \cos^2\theta)}$, this theorem is proved in § 135, Ex. 3.

The tangents at the ends of a pair of conjugate diameters, PCP' and DCD', form a parallelogram (Fig. 117), called the **conjugate parallelogram**; Theorem 6 shows that the area of this parallelogram is constant, and equal to $4ab$.

THEOREM 7.

If CP, CD are conjugate semi-diameters, then

$$CD^2 = SP \, . \, S'P.$$

Let P be the point θ; then, § 128,

$$SP = a - e(x \text{ of } P) = a - ea \cos\theta, \quad S'P = a + ea \cos\theta,$$

so that $\qquad SP \, . \, S'P = a^2 - e^2a^2 \cos^2\theta = a^2 - (a^2 - b^2)\cos^2\theta,$

and therefore $\quad SP \, . \, S'P = a^2 \sin^2\theta + b^2 \cos^2\theta = CD^2.$

THEOREM 8.

If the semi-diameter CP of an ellipse bisect the chord QQ' at V, then the tangents at Q and Q' meet at a point T on CP produced, such that

$$CV \, . \, CT = CP^2.$$

Suppose the figure (Fig. 119) projected from the corresponding figure for a circle, and use small letters to denote corresponding points on the circle. Then qq' is perpendicular to Cp, and the tangents at q and q' meet on Cp produced at t, so that

$$Cv \, . \, Ct = Cq^2 = Cp^2 \quad \text{or} \quad Cv : Cp = Cp : Ct.$$

But the ratios $Cv : Cp$ and $Cp : Ct$ are not altered by projection, since C, v, p, t lie on the same straight line; therefore (Fig. 119)

$$CV : CP = CP : CT \quad \text{or} \quad CV \, . \, CT = CP^2.$$

THEOREM 9.

If $P'CP$, $D'CD$ are two conjugate diameters of an ellipse and V the middle point of any chord QQ' parallel to CD, then
$$VQ^2 : P'V \,.\, VP = CD^2 : CP^2.$$

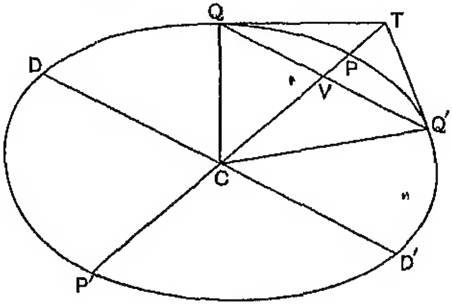


FIG. 119.

Projecting from the corresponding figure for a circle and using the same notation as in Theorem 8, we have (Fig. 119), since $d'Cd$ and $q'vq$ are perpendicular to $p'Cp$,
$$vq^2 : Cd^2 = p'v \,.\, vp : Cp^2,$$
because $vq^2 = p'v \,.\, vp$ and $Cd^2 = Cp^2$. But the ratios $vq : Cd$, $p'v : Cp$ and $vp : Cp$ are not altered by projection, because vq and Cd are on parallel lines and p', v, C, p are on the same straight line; therefore (Fig. 119)
$$VQ^2 : CD^2 = P'V \,.\, VP : CP^2$$
or
$$VQ^2 : P'V \,.\, VP = CD^2 : CP^2.$$

Since $P'V \,.\, VP = CP^2 - CV^2$, we may put the result in the form
$$\frac{CV^2}{CP^2} + \frac{VQ^2}{CD^2} = 1,$$
and, taking $P'CP$, $D'CD$ as *oblique axes*, $CV = x$, $VQ = y$, $CP = a'$, $CD = b'$, we get the equation
$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1, \quad \text{........................(1)}$$
which is the equation of the ellipse referred to two conjugate

diameters as axes. It is easy to show, by Theorem 8, that the equation of the tangent at the point (x_1, y_1) is

$$\frac{x_1 x}{a'^2}+\frac{y_1 y}{b'^2}=1. \quad \text{.......................(2)}$$

Gradients of Conjugate Diameters. If m, m' are the gradients of the conjugate diameters $P'CP$, $D'CD$ of the ellipse $x^2/a^2+y^2/b^2=1$, then

$$mm'=-\frac{b^2}{a^2}. \quad \text{.......................(3)}$$

If P is the point θ, then D is the point $\theta+\pi/2$; the coordinates of P are $a\cos\theta$, $b\sin\theta$, and those of D are $-a\sin\theta$, $b\cos\theta$. Hence

$$m=\frac{b\sin\theta}{a\cos\theta}, \quad m'=\frac{b\cos\theta}{-a\sin\theta}, \quad mm'=-\frac{b^2}{a^2}.$$

Ex. 1. If the tangent at P on an ellipse meets the directrix in Z, prove that the angle PSZ is a right angle.

Let the tangent at $P(a\cos\theta, b\sin\theta)$ be

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1. \quad \text{...........................(i)}$$

The abscissa of Z is CX or a/e; therefore, putting a/e for x in (i), we find for y, the ordinate of Z, the value

$$\frac{(e-\cos\theta)b}{e\sin\theta}.$$

Hence the gradient of SZ is

$$\frac{(e-\cos\theta)b}{e\sin\theta} \div \frac{(1-e^2)a}{e} \quad \text{or} \quad \frac{b(e-\cos\theta)}{a(1-e^2)\sin\theta}.$$

But the gradient of SP is $\dfrac{b\sin\theta}{a(\cos\theta-e)}$, and therefore the product of the gradients of SZ and SP is -1, since $b^2=a^2(1-e^2)$.

See also § 137, Theorem 2.

Ex. 2. The perpendicular from S, the focus of an ellipse, to a chord of the ellipse, meets the directrix ZX where the diameter bisecting the chord meets it.

Let the diameter meet the parallel focal chord QQ' in V, the curve in P and the directrix in T. Then, by Ex. 1 and Th. 8, TS is the perpendicular to QQ', and therefore to the given chord. This proves the proposition.

Ex. 3. To construct a pair of conjugate diameters of a given ellipse which shall contain a given angle α, and to find when the angle is a minimum.

Using the notation of Example 2, we see that angle CTS is $\left(\frac{\pi}{2}-\alpha\right)$, hence T is found by describing on CS a segment of a circle containing an angle $\left(\frac{\pi}{2}-\alpha\right)$. CT is one of the diameters, the other is the perpendicular from C to TS.

There are two positions of T; when they coincide the angle α is a minimum and the conjugate diameters are equal. It may be verified that then $\tan\frac{1}{2}\alpha=\sqrt{(1-e^2)}$.

EXERCISES XLII.

1. Find the gradient of the diameter conjugate to $y=x$.

2. A diameter of the ellipse $x^2/25+y^2/9=1$ is parallel to the line $2x+7y-5=0$; find the equation of the conjugate diameter.

3. Find the equation of the line joining the centre of the ellipse $5x^2+3y^2=15$ to the middle point of the chord whose equation is

$$x+y=1.$$

4. Establish the identity

$$(SP-CA)^2+(CA-SD)^2=CS^2,$$

following the usual notation.

5. If the diameter through a point P on an ellipse bisects the chord which is normal at Q, prove that the diameter through Q bisects the chord which is normal at P.

6. PQ is a chord of an ellipse normal at P, CZ the perpendicular from the centre C on the tangent at P, and CD the semi-diameter conjugate to CP. Prove that $PQ : 2CD=CA \,.\, CB : CD^2+PZ^2$.

7. If CP, CQ bisect chords parallel to the bisectors of the angles between the x- and y-axes, prove that the product of the gradients of CP and CQ is $-b^4/a^4$.

8. Prove that the axes form the only pair of conjugate diameters at right angles to each other.

9. If PP', a diameter of an ellipse, subtend a right angle at the point R on the ellipse, prove that the axes are parallel to RP, RP'.

10. Prove that the diameters of the ellipse $x^2/a^2+y^2/b^2=1$, whose gradients are b/a, $-b/a$, are equi-conjugate diameters.

11. If a diameter of an ellipse subtends a right angle at the ends of its conjugate, show that the length of the diameter is determined, and find the coordinates of its ends referred to the principal diameters as axes.

12. Show how to construct a pair of conjugate diameters when the angle between them is given, and prove that when the angle between a pair of conjugate diameters is a minimum, the inclination of one of them to the major axis is $\tan^{-1}\sqrt{(1-e^2)}$.

13. If CP, CD are conjugate semi-diameters and the tangents at P and D meet in T, find the locus of the middle point of PD, and also the locus of T.

14. An ellipse passes through the six points (2, 3), (3, 2), (3, 1), (1, 3), (1, 2), (2, 1); prove that its canonical equation is $\xi^2+3\eta^2-2=0$. Find the coordinates of the centre and the equations of the diameters which bisect chords parallel to the x- and y-axes.

15. If the normal at P meet the major and minor axes at G, g, prove that with the usual notation

$$(1)\ NG : CN = PG : Pg = BC^2 : AC^2,$$

$$(2)\ Sg : CD = CS : CB.$$

16. Find the equation of the chord of the ellipse $x^2/a^2+y^2/b^2=1$, whose middle point is (x_1, y_1).

17. Find the coordinates of the middle point of the chord of the ellipse $x^2/a^2+y^2/b^2=1$, whose equation is $y=mx+c$.

18. A tangent to the ellipse $x^2/a^2+y^2/b^2=1$, whose centre is C, meets the director circle $x^2+y^2=a^2+b^2$ in Q and Q'; prove that CQ and CQ' are conjugate diameters of the ellipse.

19. The locus of the centres of all ellipses which touch two given straight lines at given points is a straight line.

20. The normal at P to an ellipse meets the line joining the centre to the corresponding point on the auxiliary circle in Q; prove that

$$PQ=CD.$$

21. The perpendicular through S to CP meets the directrix where the conjugate diameter of CP meets it.

22. Prove that one pair of conjugate diameters of an ellipse is harmonically conjugate with respect to the axes, and that these diameters are equal as well as conjugate.

CHAPTER XX.

GEOMETRICAL DISCUSSION OF SECANTS, TANGENTS AND NORMALS OF A CONIC.

137. The General Conic. Some properties of the conic are most simply found by geometry, especially those relating to foci, tangents and normals. The following account of them will serve to make the student better acquainted with the curves, before proceeding to examine their properties further by analysis.

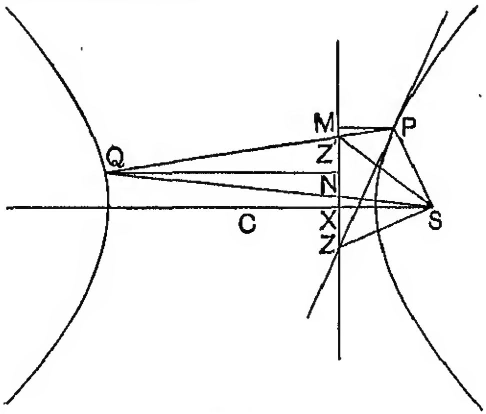


FIG. 120.

THEOREM 1.

If a secant PQ of a conic, whose focus is S, meet the directrix ZX in Z′, then Z′S bisects the angle PSQ externally or internally.

Proof. Let PM, QN (Figs. 120, 121) be perpendiculars to the directrix ZX; join SP, SQ, SZ', and produce QS, if necessary, to R.

Then $$\frac{Z'P}{Z'Q}=\frac{PM}{QN}=\frac{e\,.\,PM}{e\,.\,QN}=\frac{SP}{SQ};$$

therefore Z' cuts the base PQ of triangle PSQ externally (Fig. 121) or internally (Fig. 120) in the same ratio as the

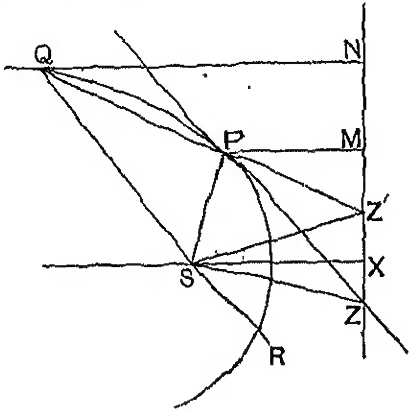


FIG. 121.

sides SP, SQ, so that $Z'S$ bisects the angle PSQ externally or internally.

THEOREM 2.

If the tangent at P on a conic, whose focus is S, meet the directrix ZX in Z, PZ subtends a right angle at the focus S.

Proof. Let the secant PQ in Fig. 121 cut the curve again in Q and the directrix in Z', and let the line PZ' turn about P till Q coincides with P; at this moment PZ' takes the position of PZ, the tangent at P. When Q is all but at P, RSP is all but two right angles, so that $Z'SP$, being half of RSP, is all but one right angle. Therefore the angle ZSP is exactly one right angle.

EXERCISES XLIII.

1. Show how to draw the tangent at P on a conic whose focus and directrix are given.

2. **If PSQ is a focal chord of a conic, the tangents at P, Q meet on the directrix.**

3. Show how to draw tangents to a conic from a point on the directrix.

4. The focal chord PSQ of a conic meets the directrix in R; prove that PQ is divided internally and externally in the same ratio at S and R.

5. If SL is the semi-latus rectum, prove that

$$\frac{2}{SL}=\frac{1}{SP}+\frac{1}{SQ},$$

where P, Q are the extremities of any focal chord.

6. PSQ, $P'SQ'$ are two focal chords of a conic. Prove that the other four lines joining P, Q, P', Q' meet in pairs in two points on the directrix which subtend a right angle at the focus.

7. PQ is a fixed focal chord of a conic and R is a variable point on the conic. RP and RQ meet the directrix in U and V; show that USV is a right angle and that $XU.XV=XS^2$.

8. PQ is a double ordinate of a conic, and the line joining P to X, the foot of the directrix, cuts the curve in P'. Show that $P'Q$ passes through the focus.

9. PSQ, a focal chord of a conic, meets the directrix in K; prove that $(PQSK)$ is a harmonic range.

10. A focal chord PSQ of a conic meets the directrix in K; prove that $\frac{2}{SK}=\frac{1}{SP}+\frac{1}{SQ}$, where SP, SQ, SK are steps on the PQ-line.

11. The segments of any focal chord of a conic subtend equal angles at the foot of the directrix.

12. PSQ, a focal chord of a conic, meets the directrix in K, the tangent at P meets the directrix in Z and the perpendicular through Q to PQ meets the tangent at P in T; prove that $Z(PQSK)$ is a harmonic pencil and that the directrix bisects QT.

13. The latus rectum cuts the tangents at the extremities of any focal chord in H and H'; prove that $SH=SH'$.

14. If the projections of K, any point on a tangent to a conic, on the directrix and the focal radius of the point of contact be I and U respectively, prove that $SU=e\,.\,KI$. **(Adams's Property.)**

15. Use Adams's Property to construct the tangents to a conic from an external point, and to show that the tangents subtend equal angles at the focus.

16. The tangent at L, the extremity of the latus rectum of a conic, meets the ordinate NP of a point P in Q; prove that $NQ=SP$.

17. Given the focus of a conic, a tangent and its point of contact, and another point on the curve, show how to construct the vertex and the directrix.

18. The tangent at P to a conic meets the directrix in Z and the axis through S in T; prove that SM touches the circle SZT.

19. If Y is the foot of the perpendicular from S to the tangent at P, prove that $SY : YX=SP : PM$, where M is the projection of P on the directrix.

138. Notation and Definitions. The following notation will be used unless the contrary is expressly stated.

S, S' are the foci of a conic, ZX and $Z'X'$ the corresponding directrices; X and X' are the feet of these directrices; A and A' are the corresponding vertices. L, L' are the extremities of the latus rectum.

The circle on AA' as diameter is called the auxiliary circle.

C is the centre of the conic; CN and NP are the abscissa and ordinate of a point P on the curve, PM the perpendicular from P on the directrix ZX.

Y, Y' are the feet of the perpendiculars from S, S' on the tangent at P.

The *normal* at P is the perpendicular at P to the tangent at P.

The tangent and normal at P meet the transverse axis of the conic in T and G respectively; NT is called the *subtangent* and NG the *subnormal* at the point P.

139. The Parabola.

THEOREM 3.

If the tangent and normal at P on a parabola be drawn, then

$$(1)\ \angle SPT=\angle MPT,\quad (2)\ SP=ST,\quad (3)\ SP=SG,$$
$$(4)\ TA=AN,\quad (5)\ NG=2AS.$$

Proof. Let the tangent at P meet the directrix in Z (Fig. 122).

(1) Angle ZSP is a right angle, by Theorem 2, so that triangles ZSP, ZMP are congruent, and $\angle SPT = \angle MPT$.

(2) By (1), $\angle SPT = \angle MPT$.

But $\angle MPT = \angle STP$,

therefore $\angle SPT = \angle STP$ and $SP = ST$.

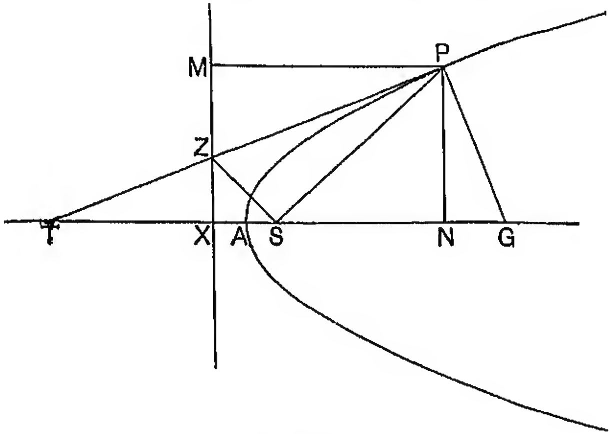


FIG. 122.

(3) By (2), S is the centre of the semi-circle which contains the right angle TPG; therefore $SP = SG$.

(4) $TA = ST - SA = SP - SA = PM - SA$.

But $PM = NX$, $SA = AX$, so that $TA = AN$.

(5) $NG = SG - SN = SP - SN = XN - SN = 2AS$.

THEOREM 4.

The locus of Y, the foot of the perpendicular from the focus on the tangent at P, is the tangent at the vertex.

Proof. Join SM (Fig. 123); then SPM is an isosceles triangle and PT bisects the vertical angle, by Theorem 3.

Therefore PT cuts SM at right angles, so that the intersection of SM and PT is Y, the foot of the perpendicular from S to PT.

Now Y, A bisect SM, SX; therefore AY, being perpendicular to AS, is the tangent at the vertex, and the locus of Y.

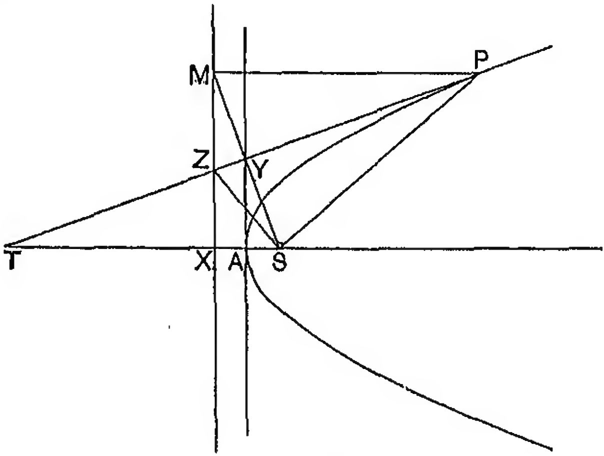


FIG. 123.

Cor. 1. SY bisects the angle TSP.

Cor. 2. $y=mx+a/m$ is the tangent of gradient m to the parabola $y^2=4ax$.

THEOREM 5.

$$SY^2=AS.SP.$$

Proof. In triangles ASY, YSP (Fig. 123),

$$\angle SAY=\angle SYP \quad \text{and} \quad \angle ASY=\angle YSP;$$

therefore the triangles are similar, and

$$AS:SY=SY:SP \quad \text{or} \quad SY^2=AS.SP.$$

THEOREM 6.

If tangents at P and P' meet in O, OP and OP' subtend equal angles at the focus, the triangles OSP and OSP' are similar, and $SO^2=SP.SP'$.

Proof. Let OP, OP' (Fig. 124) meet the tangent at the vertex in Y, Y'; join SY, SY', SO.

Then, by Theorem 4, angles SYO, $SY'O$ are right angles, so that the four points O, Y, S, Y' lie on a circle. ....(1)

Also, as in Theorem 5,

$$\angle SPY = \angle SYA = \angle SOY', \text{ by (1)},$$

and $$\angle SP'Y' = SY'A = \angle SOY, \text{ by (1)}.$$

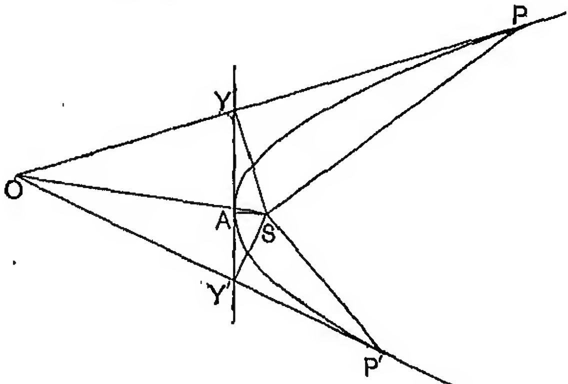


FIG. 124.

Therefore the angles SPO, SOP of triangle OSP are equal to the angles SOP', $SP'O$ of triangle OSP'.

Hence $\angle OSP = \angle OSP'$, that is, OP and OP' subtend equal angles at the focus.

Also, the triangles OSP, OSP' are similar,

and $$SO^2 = SP \cdot SP'.$$

EXERCISES XLIV.

1. Prove that $SPMT$ is a rhombus.

2. Prove that every point on PT is equidistant from S and M.

3. Given the focus and directrix and a point O on PT, show how to find P; and then give a construction for drawing the tangents from a given point O to a parabola whose focus and directrix are given.

4. Given the tangent PT and P its point of contact, given also the focus, draw the directrix.

5. Given the tangent PT but not P the point of contact, given also some point Q on the curve, and the focus, show how to find the directrix and the point P.

6. Given the tangent PT but not P the point of contact, given also the axis and the focus, show how to find the directrix and the point P.

7. Tangents at the extremities of a focal chord of a parabola intersect at right angles on the directrix.

8. The circle on a focal chord of a parabola as diameter touches the directrix.

9. Tangents at the extremities Q, Q' of a focal chord of a parabola meet at Z, and the parallel through Z to the axis meets the curve at P; prove that $QQ'=4SP$.

10. The locus of the middle points of focal chords of a parabola is a parabola.

11. If $l=SL$, the semi-latus rectum, and $SP=r$, prove that PG^2 is equal to $2lr$.

12. If PT, PG and $P'T'$, $P'G'$ be the tangents and normals at P, P', two points on a parabola, and if the difference of the squares on PG and $P'G'$ is constant, prove that TT' is constant.

13. If $l=SL$, the semi-latus rectum, and $SP=r$, prove that

$$SZ=rl/\sqrt{l(2r-l)}.$$

14. Prove that the length of the perpendicular from the focus on to the tangent at the end of the latus rectum is $AS.\sqrt{2}$.

15. Prove that XL is the tangent at L, the end of the latus-rectum.

16. Given the focus and directrix of a parabola, draw the tangent parallel to a given line.

17. If P is the point (9, 6) on the parabola $y^2=4x$, calculate PT and TN.

18. If m is the gradient of PT referred to AS and AY as axes of x and y, prove that $AY=a/m$ and $AT=a/m^2$.

19. Prove that $y=x/t+at$ is the tangent to the parabola $x=at^2$, $y=2at$ at the point t.

What is the geometrical significance of the quantity t?

20. If Y is a variable point on the line $4x-3y+1=0$ and S is the fixed point $(1, -1)$, and if YP is drawn perpendicular to SY, prove that YP envelops (that is, is a variable tangent to) the curve

$$9x^2+24xy+16y^2-122x+104y=31.$$

21. V is the middle point of QQ', a focal chord of a parabola; and the tangents at Q and Q' meet in O. If the tangent parallel to QQ' meet the curve in P and OQ, OQ' in R, R', prove that the five points O, R, R', S, V lie on a circle, centre P.

22. The external angle between the tangents OP, OP' is half the angle between SP and SP'.

23. The tangents at P and P' meet in O; prove that $\dfrac{OP^2}{OP'^2}=\dfrac{SP}{SP'}$.

24. TP, TQ are tangents from a point T to a parabola, and TS is produced to T' so that $TS=ST'$; prove that the triangles $T'SP$, $T'SQ$ are similar.

25. If TQ, TQ' be tangents from T to a parabola, the bisector of the angle QTQ' is equally inclined to ST and the axis.

26. The tangents to a parabola at P and Q intersect in T; the circles circumscribing the triangles SPT, SQT meet the axis again in H and K. Prove that PH and TK are parallel.

27. If TP, TP' are tangents to a parabola whose focus is S, show that the tangents at the points where ST cuts the parabola are parallel to the bisectors of the angle PTP'.

28. Two parabolas whose foci are S and S' have a common directrix; prove that the bisectors of the angles formed by SS' and the directrix are common tangents to the parabolas.

29. The tangents at the extremities of a focal chord of a parabola meet in T and the normals in F; prove that TF is parallel to the axis.

30. The locus of intersection of the normals at the extremities of a focal chord of a parabola is a parabola.

31. Show that the angle between any two tangents is $\cos^{-1}(r_1/r_2)$, where r_1, r_2 are the respective distances of their point of intersection from the directrix and focus.

32. The circle passing through the points of intersection of three tangents to a parabola also passes through the focus.

33. If T is a point on the latus rectum of a parabola, the tangents from T to the parabola are two of the bisectors of the angles between the latus rectum and the tangents drawn from T to the circle described on the latus rectum as diameter.

34. Given two tangents to a parabola and the focus, determine the vertex and the directrix.

35. The tangents OP, OP' are cut by a third tangent in Q, Q' respectively; prove that $OQ/QP=P'Q'/Q'O$.

36. If the normal PG be produced to meet the curve again in Q, and PQ subtend a right angle at the focus, prove that the ordinate of P is equal to the latus rectum.

37. If a parabola touches three sides of a triangle, its directrix passes through the orthocentre.

38. Prove the following construction for finding on a parabola a point P such that the portion of the tangent at P intercepted between the directrix and the tangent at the vertex is of given length L. With centre S and radius L describe a circle cutting the tangent at the vertex in B. With centre S and radius AB describe a circle cutting the tangent at the vertex in C. Draw CQ perpendicular to SC, and let it touch the parabola at Q. Find a third proportional, F, to SQ and AS. Then F is the abscissa of the required point P.

39. A circle whose centre is on the axis of a parabola touches the parabola; prove that the tangent to the circle from any point on the parabola is equal to the perpendicular let fall from the point to the chord of contact.

40. Given three tangents to a parabola, and the point of contact of one of them; determine the focus and directrix.

41. P is a variable point on a fixed line and A is a fixed point; prove that the perpendicular through P to PA envelops a fixed parabola.

42. Prove that the line joining the projections of a point on a parabola on the axis and tangent at the vertex envelops a fixed parabola.

43. Prove that the parallel through G, the foot of the normal at a point P on a parabola, vertex A, to AP touches the fixed parabola whose focus is the point $(-2a, 0)$ and whose tangent at the vertex is $x=2a$, where the axes of x and y are the axis and the tangent at the vertex of the given parabola.

44. The point P is the foot of the perpendicular from the vertex on a variable tangent, gradient m, of the parabola $y^2=4ax$; show that

$$x=-a/(1+m^2), \quad y=a/m(1+m^2)$$

are freedom-equations of the locus of P. Find the constraint-equation, and trace the locus from either of these equations. The locus is the pedal of the parabola with respect to its vertex.

140. Central Conics. The following are the important properties of the tangent and normal to a central conic, ellipse or hyperbola. The proofs refer to the ellipse, but they apply with certain obvious changes to the hyperbola.

THEOREM 7.

The focal distances SP, $S'P$ are equally inclined to the tangent and normal at P, and

$$(1)\ SG=e\,.\,SP, \quad (2)\ S'G=e\,.\,S'P, \quad (3)\ CG=e^2\,.\,CN.$$

Proof. Let the tangent at P (Fig. 125) meet the directrix in Z; join SZ, SM.

From Theorem 2, § 137, it follows that S, P, M, Z lie on a circle whose diameter is PZ; PG is the tangent at P to this circle.

Hence in triangles SPG, SPM,

$$\angle SPG = \angle SMP \quad \text{and} \quad \angle GSP = \angle SPM.$$

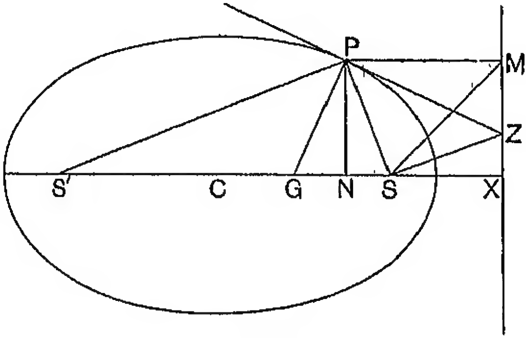


FIG. 125.

Therefore the triangles SPG, SPM are similar, and

$$SG : SP = SP : PM = e;$$

so that $SG = e \, . \, SP$; and similarly $S'G = e \, . \, S'P$.

Also, by (1) and (2), $SG : S'G = SP : S'P$,

so that the normal PG is equally inclined to SP, $S'P$.

Since the tangent is perpendicular to the normal, it also is equally inclined to SP, $S'P$.

Again, $$CG = CS - GS = e \, . \, CA - e \, . \, SP$$
$$= e^2(CX - PM) = e^2 \, . \, CN.$$

THEOREM 8.

The locus of the feet of the perpendiculars from the foci on a variable tangent is the auxiliary circle.

Proof. Let $S'P$ (Fig. 116) meet SY in H. Then, since PY bisects the angle SPH, by Theorem 7,

$$SP = PH \quad \text{and} \quad SY = YH.$$

Therefore $S'H = S'P + PH = S'P + SP = AA'$,

so that $CY = \frac{1}{2}S'H = CA$.

Hence the locus of Y is the circle, centre C, radius CA, that is, the auxiliary circle.

Note. CY is parallel to $S'P$ and CY' is parallel to SP.

Theorem 9.

$$SY \cdot S'Y' = CB^2.$$

Proof. Let $Y'S'$ meet the auxiliary circle again in Z (Fig. 116). Since $YY'Z$ is a right angle, YZ is a diameter, and therefore passes through C.

Triangles CSY and $CS'Z$ are congruent, so that $SY = S'Z$.

Therefore
$$SY \cdot S'Y' = ZS' \cdot S'Y' = AS' \cdot S'A'$$
$$= (CA + CS)(CA - CS) = CB^2.$$

Theorem 10.

(1) *If tangents at the points P and P′ on an ellipse meet in O, OP and OP′ subtend equal angles at either focus, and are equally inclined to OS and OS′, each to each.*

(2) *If tangents at the points P and P′ on a hyperbola meet in O, OP and OP′ subtend equal or supplementary angles at either focus, according as P and P′ are on the same branch or on opposite branches of the hyperbola; also OP and OP′ are inclined at equal or supplementary angles to OS and OS′, each to each, according as P and P′ are on opposite branches or on the same branch of the hyperbola.*

Proof. Produce SY, $S'Y'$ in Fig. 126 to meet $S'P$, SP' in H, H'; join OH, OH'.

Then, as in Theorem 8,

$$SP = HP \quad \text{and} \quad S'H = AA'.$$

Hence, in triangles SPO and HPO,

$$SP = HP, \quad OP = OP, \quad \angle SPO = \angle HPO,$$

because PO, the tangent at P, is equally inclined to PS and PS'; therefore

$$\angle OSP = \angle OHP, \quad \angle POS = \angle POH,$$

and $$OS = OH.$$

Similarly, $\angle OS'P' = \angle OH'P'$, $\angle P'OS' = \angle P'OH'$,

and $$OS' = OH'.$$

Now, in triangles $OS'H$ and $OH'S$,

$$OH = OS, \quad OS' = OH', \quad S'H = AA' = H'S;$$

therefore $\angle S'OH = \angle H'OS$, so that $\angle SOH = \angle S'OH'$.

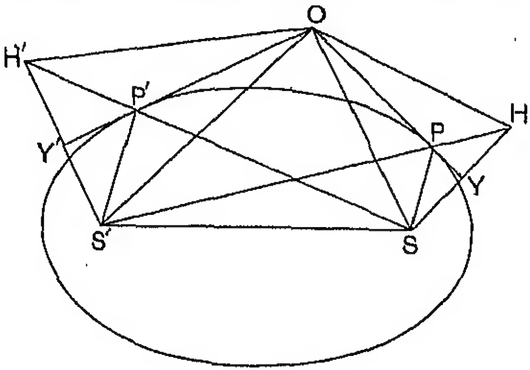


FIG. 120.

But it was shown that $\angle POS = \angle POH$

and $$\angle P'OS' = \angle P'OH';$$

therefore $$\angle POS = \angle P'OS',$$

or OP, OP' are equally inclined to OS, OS'.

Also, from the congruency of triangles $OS'H$, $OH'S$,

$$\angle OHS' = \angle OSH',$$

and it was proved that

$$\angle OSP = \angle OHP;$$

therefore $$\angle OSP = \angle OSP';$$

that is, OP and OP' subtend equal angles at the focus.

The Asymptotes as Tangents. Since the asymptotes are tangents, Theorems 8 and 9 show that the feet of the perpendiculars from the foci on the asymptotes lie on the auxiliary circle, and that the length of each perpendicular is the semi-conjugate axis CB.

If the tangent at P on a hyperbola meet the asymptotes in T and T', as in Fig. 127, and SK, SK' are parallel to the asymptotes CT, CT', then ST bisects the angle PSK, so that

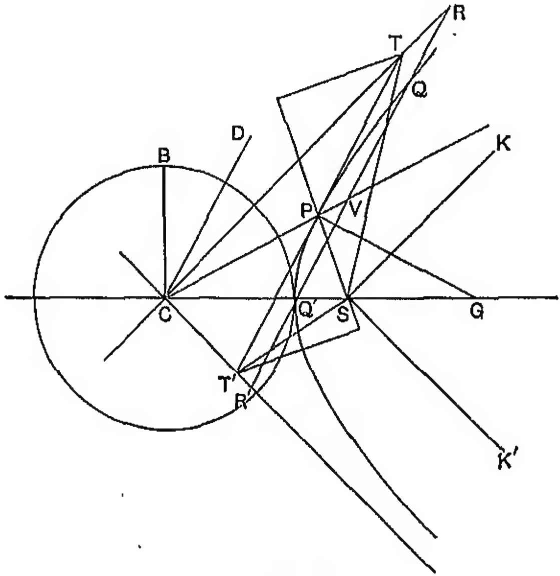


FIG. 127.

T is equidistant from SP and SK. But the perpendicular from T to SK is equal to the perpendicular from S to CT, which, as remarked above, is equal to CB. Hence the perpendicular from T to SP is equal to CB. Similarly, the perpendicular from T' to SP, being equal to the perpendicular from T' to SK', is equal to CB. Hence T and T' are equidistant from SP, so that P bisects TT' (see Th. 6, p. 336).

Further, CP bisects chords parallel to TT' (§ 132), so that

if CD is conjugate in direction to CP, CD is parallel to TT'. Hence (§ 45) $C(PDTT'')$ is a harmonic pencil, or the asymptotes are harmonically conjugate with respect to CP, CD.

If the hyperbola is rectangular, TPC is an isosceles triangle, so that the conjugate directions CP, CD are equally inclined to each of the asymptotes.

141. Worked Examples. We shall now work some examples of the application of the above Theorems.

Ex. 1. SP, $S'Q$ are focal radii of a conic which are parallel and in the same direction; prove that the tangents at P and Q meet on the auxiliary circle.

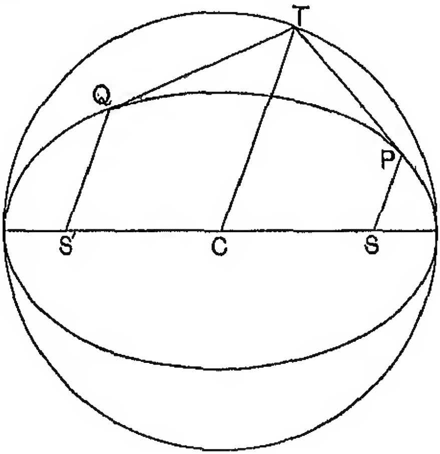


FIG. 128.

Draw CT (Fig. 128) parallel to SP or $S'Q$ and in the same direction to meet the auxiliary circle in T. Then the tangents at P and Q pass through T, according to Theorem 8, *Cor.*

Ex. 2. If the normal at P on a rectangular hyperbola meet the transverse axis in G, then $CP=PG$.

Draw the tangent at P as in Fig. 127 to meet the asymptotes in T and T''; draw CD parallel to PT.

P is the middle point of the hypotenuse of triangle TCT''; therefore $\angle PCT = \angle PTC = \angle TCD$.

Hence CT bisects both $\angle PCD$ and $\angle GCB$, so that $\angle PCG = \angle BCD$.

Now PG, GC are perpendicular to CD, CB; therefore $\angle PGC = \angle BCD$.

Hence $$\angle PCG = \angle PGC \text{ and } CP = PG.$$

Ex. 3. PSQ is a focal chord of a conic; the normals at P, Q intersect in O and the tangents in Z; prove that OZ passes through S'.

The four points Z, P, O, Q in Fig. 129 lie on a circle, Z lies on the directrix and ZS is perpendicular to PQ.

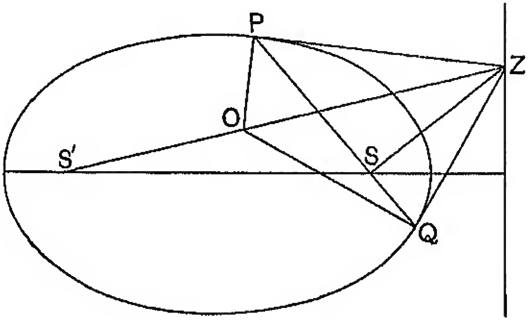


FIG. 129.

Hence $\angle QZS$ = complement of $\angle ZQS = \angle OQP = \angle OZP$.

Therefore ZO and ZS make equal angles with the tangents ZP and ZQ, so that ZO passes through S', by Theorem 10.

EXERCISES XLV.

1. If the parallel through C to the tangent at P meet SP, $S'P$ at E, E', prove that $PE = PE' = CA$.

2. Given the focus, directrix and a tangent of a conic, show how to determine its centre.

3. If P is a point on an ellipse, whose foci are S and S', prove that the in-centre of the triangle SPS' divides the normal PG in the ratio $1 : e$, where e is the eccentricity.

4. If TQ and TQ' are tangents to a conic, the bisectors of angles QTQ' and STS' coincide.

5. E and F are points on a tangent to a conic, whose focus is S, such that ESF is a right angle. The other tangents from E and F to the conic meet it at P and Q; prove that PQ is a focal chord.

6. Prove that the external angle between two tangents to an ellipse is half the sum of the angles subtended at the foci by the chord of contact.

7. From a movable point Z on the directrix of a conic, a tangent is drawn which meets the major axis in T. Show that the locus of the intersection of the other tangents from Z and T to the conic is a straight line perpendicular to the major axis.

8. Perpendiculars SY, SZ are drawn from the focus S of an ellipse to two tangents TP, TQ meeting them in Y and Z. Prove that YZ is at right angles to $S'T$.

9. If Y is the foot of the perpendicular from S to the tangent at P, prove that $SY : YX = SG : SP$.

10. If the tangent and normal at a point P on an ellipse meet the major axis at T and G, prove that $CG \,.\, CT = CS^2$.

11. A is a fixed point; P is a variable point on a fixed circle; PQ is drawn perpendicular to AP; prove that PQ envelops a conic which is a hyperbola or an ellipse, according as A lies outside or inside the circle.

12. A variable circle is drawn through a fixed point so as to have the same radius as a fixed circle; prove that the common chord of the fixed and variable circles envelops a conic which is a hyperbola or an ellipse, according as the fixed point lies outside or inside the fixed circle.

13. If the tangent at P meet the directrix in T, and $l = SL$, $r = SP$, e = eccentricity, prove that

$$ST = lr/\sqrt{\{(l - \overline{1-e} \,.\, r)(\overline{1+e} \,.\, r - l)\}}.$$

14. If $l = SL$ and $SP = r$, prove that

$$PG^2 = 2lr - (1 - e^2)r^2.$$

15. If PSQ, $P'S'Q'$ are parallel focal chords of an ellipse, prove that the intersections of the tangents at P, P', Q, Q' lie on a directrix or on the auxiliary circle.

16. P is any point on an ellipse, PSQ is a focal chord and PCP' is a diameter; prove that the tangents to the ellipse at Q and P' meet on the auxiliary circle.

17. T is a point on the auxiliary circle of an ellipse, TP and TQ are the tangents from T to the ellipse; prove that the focal distances of T are at right angles to TP and TQ.

18. The line through O, the intersection of the normals at the extremities of a focal chord PP', parallel to SS', bisects PP'.

19. If O is the intersection of the normals PG, $P'G'$ at the extremities of a focal chord PP' and OH parallel to PP' meets the axis in H, then H is the middle point of GG'.

20. If O, Z are the intersections of the normals and tangents at the extremities of a focal chord PSP', and if D, E are the projections of O, Z on PP' respectively, prove that $PD = P'E$.

21. If O, Z are the intersections of the normals and tangents at the extremities of a focal chord PSP', prove that the line joining O to the orthocentre of the triangle ZPP' is parallel to AA'.

22. If M is the projection of P on the directrix of an ellipse, prove that the locus of the intersection of SM and PG is the line BB'.

23. The normal at P to an ellipse meets the major axis in G and the minor axis in g; prove that PG/Pg is constant, and that Sg is a mean proportional between Pg and Gg.

24. The circle through the foci and any point on an ellipse passes through the intersections of the minor axis with the tangent and normal at the point.

25. If the normal at P meet the minor axis in g, and n be the projection of P on the minor axis, prove that $Cg/Cn = CS/SX$.

26. The tangent at P, a point on an ellipse, meets the minor axis in t, and the projection of P on the minor axis is n; prove that $Cn \cdot Ct = CB^2$.

27. The normal PG meets CF, the parallel to the tangent at P, in F; prove that $PG \cdot PF = CB^2$. If PG meets the minor axis in g, prove that $Pg \cdot PF = CA^2$.

28. Prove that the projection of PG on SP or $S'P$ is equal to the semi-latus rectum.

29. Find an expression for the subnormal of a central conic in terms of the central abscissa, and deduce the corresponding theorem for the parabola.

30. Express the subtangent of an ellipse in terms of the central abscissa; and deduce that, for a parabola, the subtangent is twice the abscissa measured from the vertex.

31. A circle has its centre on the major axis of an ellipse and touches the ellipse at Q and R; show that, if P is a variable point on the ellipse, the length of the tangent from P to the circle bears a constant ratio to the perpendicular from P to QR.

32. Tangents are drawn to an ellipse from any point T on the auxiliary circle. Show that the perpendicular drawn through one of the foci, S, to ST is parallel to one of the tangents and meets the other on a fixed straight line which is at right angles to the axis and cuts it at K, where $CK^2 - SK^2 = CA^2$.

33. PG is the normal to a conic at P, and L is the projection of G on SP; the line GN is drawn parallel to $S'P$ to meet the tangent at P in N, and LR and $S'Z$ are drawn perpendicular to the tangent at P. Prove that $PN/PR = S'P^2/S'Z^2$.

34. PG is the normal at a point P of an ellipse. If BCB' is the minor axis and MQ the ordinate of a point Q, such that $BM = CP$, prove that AG, $A'G$ are equal to the focal distances of Q.

35. The four focal radii drawn to any two points of an ellipse have one common tangential circle whose centre is the intersection of the tangents at the extremities of the radii.

36. An ellipse rolls on an equal ellipse. If extremities of the major axes are initially in contact, find the locus of either focus of the rolling ellipse.

37. A variable ellipse touches a fixed ellipse and has a common focus with it; find the locus of its other focus when its major axis is given.

38. Given one asymptote of a rectangular hyperbola and two points on the curve, find the centre.

39. Prove that the intersections of the directrices and the auxiliary circle of a hyperbola lie on one or other of the asymptotes.

40. If the parallel to an asymptote of a hyperbola through the point P on the curve meet the directrices in M and M', prove that PM and PM' are equal to the focal distances of P.

CHAPTER XXI.

ANALYTICAL DISCUSSION OF SECANTS, TANGENTS AND NORMALS TO CONICS.

142. The Parabola. Let

$$y^2 = 4ax \quad \text{.........................(1)}$$

and

$$x = at^2, \quad y = 2at \quad \text{.....................(2)}$$

be the constraint- and freedom-equations of a parabola.

Then

$$(y - y_1)(y - y_2) = y^2 - 4ax \quad \text{................(3)}$$

is the equation of the secant which cuts the curve at the points (x_1, y_1), (x_2, y_2). For (3) reduces to a linear equation in x, y and is satisfied by $x = x_1$, $y = y_1$ and by $x = x_2$, $y = y_2$.

Put $y_2 = y_1$ and $y_1^2 = 4ax_1$ in (3); then, after reduction,

$$yy_1 = 2a(x + x_1); \quad \text{......................(4)}$$

this is the equation of **the tangent at the point (x_1, y_1).**

If $x_1 = at^2$, $y_1 = 2at$, then (4) becomes

$$y = \frac{x}{t} + at, \quad \text{.........................(5)}$$

which is the equation of **the tangent at the point t.**

If $t = \frac{1}{m}$, (5) becomes $\quad y = mx + \frac{a}{m}, \quad$ ....(6)

which gives the equation of **the tangent of gradient m.** It follows that $y = mx + c$ is a tangent to (1) if $c = a/m$.

We also see that $t = \cot\theta$, where θ is the angle between the axis and the tangent at the point t, so that the freedom-equations of a parabola may be written

$$x = a\cot^2\theta, \quad y = 2a\cot\theta. \quad \text{.................(7)}$$

The normal at the point t is, from (5),

$$(y-2at)+t(x-at^2)=0$$

or $$y+tx=2at+at^3. \quad\text{.....................(8)}$$

If $t=-m$, (8) becomes

$$y=mx-2am-am^3, \quad\text{.....................(9)}$$

which is the equation of the normal whose gradient is m.

Equation (5) may be written as a quadratic in t, thus.

$$at^2-ty+x=0; \quad\text{.....................(10)}$$

if x, y are regarded as known there are two values of t to correspond; these values give the points of contact of tangents to the curve from the known point (x, y).

Equation (8) may be written as a cubic in t, thus:

$$at^3+t(2a-x)-y=0, \quad\text{.................(11)}$$

showing that *three* normals can be drawn to the curve from a known point (x, y); the feet of the normals are given by the roots of the cubic. One root of the cubic must be real, so that one real normal can be drawn from any point to the curve. If the three roots of the cubic (11) are real, then, by § 106,

$$4\frac{(2a-x)^3}{a^3}+27\frac{y^2}{a^2} \lesseqgtr 0$$

or $$27ay^2 \lesseqgtr 4(x-2a)^3. \quad\text{.....................(12)}$$

When $27ay^2=4(x-2a)^3$, two of the normals are coincident. When the feet of two of the normals from a point C coincide at P, C is called the **centre of curvature**, the circle, with centre C and radius CP, is called the **circle of curvature**, and CP is called the **radius of curvature** at P. The locus of the centre of curvature is called the **evolute** of the original curve. The circle of curvature meets the curve at three coincident points, and therefore lies as close to the curve at the point as a circle can lie. The centre of curvature is often spoken of as *the intersection of consecutive normals.*

EXERCISES XLVI.

1. Prove that the equation of the chord of the parabola $y^2=4ax$, whose extremities are the points t_1 and t_2, may be written as

$$(t_1+t_2)y-2x=2at_1t_2.$$

2. If the chord of the parabola $x=at^2$, $y=2at$, whose extremities are the points t_1 and t_2, is the normal at the point t_1, prove that

$$t_1+t_2=-\frac{2}{t_1}.$$

Hence show that the other extremity of the normal at the point t is the point $-t-2/t$.

3. Prove that the tangents at the points t_1, t_2 intersect at the point whose coordinates are

$$\{at_1t_2,\quad a(t_1+t_2)\},$$

and the normals at the point

$$\{a(t_1^2+t_1t_2+t_2^2+2),\quad -at_1t_2(t_1+t_2)\}.$$

4. If the feet of two of the three normals from a point C to the parabola $y^2=4ax$ coincide at the point t, prove that the coordinates of the point C are $\qquad a(3t^2+2),\quad -2at^3.$

Find the radius of curvature at the point t, and the equation of the evolute.

5. If the normal at $P(at^2, 2at)$ to the parabola $y^2=4ax$ meets the parabola again in Q, and A is the vertex, prove that the area of the triangle AQP is $\qquad 2a^2(1+t^2)(2+t^2)/t.$

6. If the normal $y=-tx+2at+at^3$ to the parabola $y^2=4ax$ subtend a right angle at the vertex, determine the value of t.

7. Find the values of m so that $y=mx+a/m$ may be a tangent to the two parabolas $y^2=4ax$ and $y^2=4b(x+c)$.

8. Chords of the parabola $y^2=4ax$ are drawn to touch the parabola $y^2=4bx$; show that the locus of the intersection of tangents at their extremities is the parabola $by^2=4a^2x$.

9. If the straight line $y=mx+c$ touches the parabola $y^2=4a(x+a)$, prove that $c=a\left(m+\dfrac{1}{m}\right)$.

10. Show that the tangents to the circle $x^2+y^2=a^2$ at the points where the straight line $x+h=0$ cuts it are also tangents to the parabola $y^2=4h(x+h)$.

11. Show that if the normal at P to a parabola meets the curve again at Q, and U is the middle point of PQ, the product of the ordinates of P and U is constant.

12. If the normal at the point P on a parabola cut the axis in G, the length of the chord drawn through G parallel to the tangent at P is equal to $4\sqrt{2}\,.\,SP$.

13. Show that the tangent to the parabola $y^2=4ax$ at the point where the normal parallel to $y+mx=0$ meets the curve again is

$$my(2+m^2)+m^2x+a(2+m^2)^2=0.$$

14. From a fixed point P on the parabola $y^2=4ax$, chords PQ, PQ' are drawn making equal angles ϕ with the tangent at P. Show that QQ' will for all values of ϕ pass through the same point R. Prove further that if P moves along the parabola, the locus of R is

$$(x+2a)y^2+4a^3=0.$$

15. Write down the coordinates of any point upon the parabola $y^2+4k(y-x)=0$ in terms of a single parameter.

16. If the tangents at two points of a parabola meet at (x, y) and the normals at (ξ, η), then $a\eta+xy=0$, where $4a$ is the latus rectum.

17. Prove that the parabola $y^2=4ax$ may be defined as the locus of a point P such that OP^2 is proportional to $PM \cdot PN$, where O is a fixed point on the parabola and PM, PN are the perpendiculars from P on two fixed straight lines, one of which is the tangent to the parabola at O and the other a tangent to the parabola $y^2=4a(x+4a)$.

18. Find the equation of the normal to the parabola $y^2=4ax$, which makes an angle θ with the axis of x. From any point in this normal two other normals are drawn to the curve. Prove that the straight line joining their feet is parallel to a fixed straight line.

19. Find the equations of the two real common tangents to the curves

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{and} \quad y^2=2lx.$$

20. Two normals to a parabola are at right angles and meet the axis in G and G'; show that the semi-latus rectum is a harmonic mean between the distances of G and G' from the focus.

21. P, Q, R, S are the vertices in order of a variable rectangle. P and R lie on the x-axis, Q on the y-axis, and P is fixed. Prove (1) that the locus of S is a parabola, (2) that QR touches a second parabola, (3) that RS is normal to a third parabola.

22. Prove that the locus of points at which a parabola subtends a given angle $(\pi-\alpha)$ is a hyperbola with the same focus and directrix and an eccentricity $\sec\alpha$.

23. Show that the line $y=mx+m(c-2a)-am^3$ is a normal to the parabola $y^2=4a(x+c)$. Prove that, if $a>b>0$ and $c>2(a-b)$, the two parabolas $y^2=4a(x+c)$, $y^2=4bx$ have a pair of common normals inclined to the common axis, and that the distance d between the curves measured along one of these common normals is given by

$$d^2=4(a-b)(c-a+b).$$

24. The area of the triangle formed by the three tangents drawn at the points (x_1, y_1), (x_2, y_2), (x_3, y_3), on the parabola $y^2=4ax$ is

$$(y_1-y_2)(y_2-y_3)(y_3-y_1)/16a.$$

25. Prove that perpendicular normal chords of a parabola divide one another in the ratio 3 : 1.

26. From a point T on the latus rectum of a parabola two tangents are drawn to the curve, and the corresponding normals intersect in G. Show that the middle point of TG lies on the axis of the parabola.

27. Show that the locus of the intersection of the normals at the ends of a system of parallel chords of a parabola is a straight line which is a normal to the curve.

28. Find the condition that the line $lx+my+n=0$ may touch the parabola of which the focus is at the origin and the vertex at the point $(a, 0)$. Show that if the two parabolas

$$y^2=4a(x-f) \quad \text{and} \quad x^2=4b(y-g)$$

touch one another, then

$$(fg-9ab)^2=4(f^2+3bg)(g^2+3af).$$

29. If normals PO, QO to a parabola intersect at right angles in O, the third normal RO through the point O is cut by the axis in a point G, such that $3OG=OR$.

30. The normals at two points P and Q on the parabola $y^2=4ax$ intersect on a fixed diameter $y=k$; prove that the tangents at P and Q to the parabola intersect on the hyperbola $xy+ak=0$.

31. The normal at P to a parabola meets the curve again in Q, and the tangents at P and Q meet in T. Show that the minimum value of the area of the triangle TPQ is twice the square on the latus rectum.

32. If two normals of the parabola $y^2=4ax$ make complementary angles with the axis, show that their point of intersection lies on one of the curves $y^2=a(x-a)$, $y^2=a(x-3a)$.

33. The normal at P to the parabola $y^2=4ax$ meets the axis in R and the parabola again in Q; the normal at Q meets the axis in R'. A line RS equal to RR' is drawn through R perpendicular to the axis; show that the locus of S is

$$(x-2a)y=4a(x-a).$$

34. Tangents are drawn to the parabola $y^2=4ax$ from the point (x', y'); show that the corresponding normals intersect in the point

$$\left(2a-x'+\frac{y'^2}{a},\ -\frac{x'y'}{a}\right).$$

35. A parabola whose axis is along the axis of x intersects the ellipse $x^2/a^2+y^2/b^2=1$ orthogonally at the point whose eccentric angle is ϕ. Show that the latus rectum of the parabola is $2a\sin^2\phi/\cos\phi$.

36. Find the coordinates of the feet of the normals from the point $(\frac{27}{4}a,\ -\frac{15}{4}a)$ to the parabola $y^2=4ax$.

143. The Ellipse. Let

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \;\ldots\ldots(1) \quad \text{and} \quad x=a\cos\theta,\; y=b\sin\theta \;\ldots\ldots(2)$$

be the constraint- and freedom-equations of an ellipse.

Then

$$\frac{(x-x_1)(x-x_2)}{a^2}+\frac{(y-y_1)(y-y_2)}{b^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1 \;\ldots\ldots(3)$$

is the equation of the chord whose ends are $(\mathbf{x}_1, \mathbf{y}_1)$, $(\mathbf{x}_2, \mathbf{y}_2)$; because (3) reduces to a linear equation in x, y, and is satisfied when $x=x_1$, $y=y_1$ and when $x=x_2$, $y=y_2$.

Put $x_1=a\cos\theta_1$, $y_1=b\sin\theta_1$, $x_2=a\cos\theta_2$, $y_2=b\sin\theta_2$ in (3); then, after simplification, we get

$$\frac{x}{a}\cos\frac{\theta_1+\theta_2}{2}+\frac{y}{b}\sin\frac{\theta_1+\theta_2}{2}=\cos\frac{\theta_1-\theta_2}{2}; \;\ldots\ldots\ldots(4)$$

this is the equation of the chord whose ends are θ_1, θ_2.

Put $x_2=x_1$ and $y_2=y_1$ in (3); then, after reduction,

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1; \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

this is the equation of the tangent at $(\mathbf{x}_1, \mathbf{y}_1)$.

Put $x_1=a\cos\theta$, $y_1=b\sin\theta$ in (5); then

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

is the equation of the tangent at the point θ.

The equation of the normal at $(\mathbf{x}_1, \mathbf{y}_1)$ is, from (5),

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

The equation of the normal at the point θ is, from (6),

$$(x-a\cos\theta)\frac{a}{\cos\theta}-(y-b\sin\theta)\frac{b}{\sin\theta}=0$$

or

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2. \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Since $\sin\theta=\frac{2t}{1+t^2}$ and $\cos\theta=\frac{1-t^2}{1+t^2}$,

where $t=\tan\frac{\theta}{2}$, we may use (see § 89) as freedom-equations, instead of (2),

$$x=\frac{a(1-t^2)}{1+t^2},\quad y=\frac{2bt}{1+t^2}.\ \ldots\ldots\ldots\ldots(9)$$

Equation (6) then becomes

$$\left(1+\frac{x}{a}\right)t^2-\frac{2yt}{b}+1-\frac{x}{a}=0,\ \ldots\ldots\ldots\ldots(10)$$

the equation of the **tangent at the point t.** If (x, y) be regarded as known, then (10) is a quadratic in t, whose roots give the points of contact of the two tangents from (x, y).

Equation (8) becomes

$$byt^4+2(ax+a^2-b^2)t^3+2(ax-a^2+b^2)t-by=0,\ \ldots(11)$$

the equation of the **normal at the point t.** If (x, y) be regarded as known, then (11) is a quartic in t, whose roots give the feet of the four normals drawn from (x, y) to the ellipse.

It is easily shown (§ 135, Ex. 1) that

$$y=mx+c$$

is a tangent to the ellipse if

$$c=\pm\sqrt{a^2m^2+b^2},\ \ldots\ldots\ldots\ldots(12)$$

and that $$lx+my=n$$

is a tangent if $$a^2l^2+b^2m^2=n^2.\ \ldots\ldots\ldots\ldots(13)$$

144. Worked Examples. We shall now work some examples on the ellipse.

Ex. 1. If the normals at the four points θ_1, θ_2, θ_3, θ_4 on the ellipse are concurrent, prove that

$$\theta_1+\theta_2+\theta_3+\theta_4=(2n+1)\pi.$$

From equation (11) it follows that $T_2=0$ and $T_4=-1$, where T_2 means the sum of the products of $\tan\frac{\theta_1}{2}$, $\tan\frac{\theta_2}{2}$, $\tan\frac{\theta_3}{2}$, $\tan\frac{\theta_4}{2}$ taken two at a time; and so on.

But $$\tan\tfrac{1}{2}(\theta_1+\theta_2+\theta_3+\theta_4)=\frac{T_1-T_3}{1-T_2+T_4}=\infty;$$

therefore $$\theta_1+\theta_2+\theta_3+\theta_4=(2n+1)\pi.$$

Ex. 2. If the normals to the ellipse at θ_1, θ_2, θ_3 are concurrent,
$$\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_1)+\sin(\theta_1+\theta_2)=0,$$
and conversely.

Let θ_4 be the foot of the fourth normal from the point of concurrency of the three specified normals. Then, as in Ex. 1, $T_3=0$; therefore
$$\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}+\tan\frac{\theta_3}{2}\tan\frac{\theta_1}{2}+\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}$$
$$=-\tan\frac{\theta_4}{2}\left(\tan\frac{\theta_1}{2}+\tan\frac{\theta_2}{2}+\tan\frac{\theta_3}{2}\right)$$
$$=\frac{\tan\frac{\theta_1}{2}+\tan\frac{\theta_2}{2}+\tan\frac{\theta_3}{2}}{\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}},\text{ since } T_4=-1,$$
$$=\cot\frac{\theta_2}{2}\cot\frac{\theta_3}{2}+\cot\frac{\theta_3}{2}\cot\frac{\theta_1}{2}+\cot\frac{\theta_1}{2}\cot\frac{\theta_2}{2}.$$

Therefore $$\Sigma\left(\cot\frac{\theta_2}{2}\cot\frac{\theta_3}{2}-\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}\right)=0,$$

that is, $$\Sigma\frac{2(\cos\theta_2+\cos\theta_3)}{\sin\theta_2\sin\theta_3}=0$$

or $$\Sigma\sin\theta_1(\cos\theta_2+\cos\theta_3)=0$$

or $$\Sigma\sin(\theta_2+\theta_3)=0.$$

Since the steps are reversible, the converse holds.

Ex. 3. If the normals at four points on the ellipse are concurrent and two of the points lie on the line
$$\frac{lx}{a}+\frac{my}{b}+1=0,$$
the other two will lie on the line
$$\frac{x}{al}+\frac{y}{bm}-1=0.$$

Let θ_1, θ_2, θ_3, θ_4 be the four points. Then
$$\frac{x}{a}\cdot\frac{\cos\frac{1}{2}(\theta_1+\theta_2)}{\cos\frac{1}{2}(\theta_1-\theta_2)}+\frac{y}{b}\cdot\frac{\sin\frac{1}{2}(\theta_1+\theta_2)}{\cos\frac{1}{2}(\theta_1-\theta_2)}-1=0, \quad\text{.................(i)}$$
$$\frac{x}{a}\cdot\frac{\cos\frac{1}{2}(\theta_3+\theta_4)}{\cos\frac{1}{2}(\theta_3-\theta_4)}+\frac{y}{b}\cdot\frac{\sin\frac{1}{2}(\theta_3+\theta_4)}{\cos\frac{1}{2}(\theta_3-\theta_4)}-1=0 \quad\text{.................(ii)}$$
are the equations of a pair of chords joining the four points.

Now $$\frac{\cos\frac{1}{2}(\theta_1+\theta_2)}{\cos\frac{1}{2}(\theta_1-\theta_2)}=-\frac{\cos\frac{1}{2}(\theta_3-\theta_4)}{\cos\frac{1}{2}(\theta_3+\theta_4)}$$

if $\quad 2\cos\frac{1}{2}(\theta_1+\theta_2)\cos\frac{1}{2}(\theta_3+\theta_4)+2\cos\frac{1}{2}(\theta_3-\theta_4)\cos\frac{1}{2}(\theta_1-\theta_2)=0,$

that is, if $\quad \cos\frac{1}{2}(\theta_1+\theta_2+\theta_3+\theta_4)+\cos\frac{1}{2}(\theta_1+\theta_2-\theta_3-\theta_4)$
$$+\cos\frac{1}{2}(\theta_3-\theta_4+\theta_1-\theta_2)+\cos\frac{1}{2}(\theta_3-\theta_4-\theta_1+\theta_2)=0,$$

that is, if $\{\sin(\theta_3+\theta_4)+\sin(\theta_4+\theta_2)+\sin(\theta_2+\theta_3)\}=0$, by Ex. 1.

And this is true, by Ex. 2.

Hence (i) and (ii) may be written in the form

$$\frac{lx}{a}+\frac{my}{b}+1=0,\quad \frac{x}{al}+\frac{y}{bm}-1=0.$$

145. The Hyperbola. Let

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \,\ldots\ldots(1) \quad \text{and} \quad x=a\sec\theta,\ y=b\tan\theta \,\ldots\ldots(2)$$

be the constraint- and freedom-equations of a hyperbola.

Then

$$\frac{(x-x_1)(x-x_2)}{a^2}-\frac{(y-y_1)(y-y_2)}{b^2}=\frac{x^2}{a^2}-\frac{y^2}{b^2}-1 \,\ldots\ldots(3)$$

is the equation of the chord whose ends are (x_1, y_1), (x_2, y_2).

The equation of the chord joining the two points θ_1, θ_2 on the curve is

$$\frac{x}{a}\cos\frac{\theta_1-\theta_2}{2}-\frac{y}{b}\sin\frac{\theta_1+\theta_2}{2}=\cos\frac{\theta_1+\theta_2}{2}. \,\ldots\ldots\ldots\ldots(4)$$

The equation of the tangent at the point (x_1, y_1) is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1, \,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

and of the tangent at the point θ,

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1. \,\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

The equation of the normal at the point (x_1, y_1) is

$$\frac{x-x_1}{\dfrac{x_1}{a^2}}+\frac{y-y_1}{\dfrac{y_1}{b^2}}=0, \,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

and of the normal at the point θ,

$$ax\sin\theta+by=(a^2+b^2)\tan\theta. \,\ldots\ldots\ldots\ldots\ldots(8)$$

The line $$y=mx+c$$
is a tangent if $$c=\pm\sqrt{a^2m^2-b^2}\,;\qquad(9)$$
and $$lx+my=n$$
is a tangent if $$a^2l^2-b^2m^2=n^2.\qquad(10)$$

EXERCISES XLVII.

1. Prove that the point of intersection of tangents at the points θ_1, θ_2 on the ellipse $x^2/a^2+y^2/b^2=1$ has coordinates
$$a\frac{\cos\frac{1}{2}(\theta_1+\theta_2)}{\cos\frac{1}{2}(\theta_1-\theta_2)},\quad b\frac{\sin\frac{1}{2}(\theta_1+\theta_2)}{\cos\frac{1}{2}(\theta_1-\theta_2)}.$$

2. Prove that the eccentric angles θ_1, θ_2 of the ends of any chord of the ellipse $x^2/a^2+y^2/b^2=1$, which is parallel to the tangent at the point θ, satisfy the relation $\theta_1+\theta_2=2\theta$.

3. If the chords joining the pairs of points θ, θ_1 and θ, θ_2 are perpendicular, prove that
$$\tan\frac{\theta+\theta_1}{2}\tan\frac{\theta+\theta_2}{2}=-\frac{b^2}{a^2}.$$

4. Prove that the point $\left(\dfrac{a^2-b^2}{a^2+b^2}a\cos\theta,\ \dfrac{b^2-a^2}{a^2+b^2}b\sin\theta\right)$ lies on the normal at the point θ. Prove also that every chord through the first point subtends a right angle at the second point.

5. Prove that the feet of the normals to the ellipse $x^2/a^2+y^2/b^2=1$, which meet at the point (h, k), lie on the rectangular hyperbola
$$(a^2-b^2)xy-a^2hy+b^2kx=0.$$

6. P is a point whose projections on the major and minor axes of an ellipse are the points in which these axes are cut by a normal; show that the locus of P is an ellipse.

7. Prove that the tangents drawn at the points θ, $\theta+\dfrac{2\pi}{3}$, $\theta-\dfrac{2\pi}{3}$ on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ intersect in pairs on the ellipse $x^2/a^2+y^2/b^2=4$, and that the centroid of the triangle formed by the tangents is the common centre of the ellipses.

8. Prove that a one-fold infinity of triangles can be inscribed in an ellipse such that the centroid of each coincides with the centre of the conic. If PQR be such a triangle and $P'Q'R'$ the triangle formed by the tangents which touch the conic at P, Q, R, show that the centroid of triangle $P'Q'R'$ also coincides with the centre of the conic.

9. Find the intersection of the normals at the points θ, ϕ on the ellipse $x^2/a^2+y^2/b^2=1$, and deduce the point of intersection of "consecutive normals" (centre of curvature) at the point θ. Find also the radius of curvature, and prove that the equation of the evolute is

$$(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}.$$

10. Normals at P, Q on the ellipse $x^2/a^2+y^2/b^2=1$ meet the major axis in G, K respectively: prove that the projections of PG, QK on PQ are equal, and deduce (geometrically) that $TP/TQ=PG/QK$, where T is the intersection of the tangents at P, Q.

If the common value of the projection is q and if $PQ=d$, prove that $q/d=\frac{1}{2}b^2/k^2$, where k is the semi-diameter parallel to PQ, and b is the semi-minor axis.

11. Prove that the locus of the in-centre of triangle PSS' as P moves round an ellipse, whose foci are S, S' and whose eccentricity is e, is an ellipse whose major axis is SS' and whose eccentricity is $[2e/(1+e)]^{\frac{1}{2}}$.

12. P is any point $(a\cos\theta, b\sin\theta)$ on an ellipse and PSQ, $PS'R$ are focal chords. Prove that the distance of P from QR is

$$\frac{2b(1-e^2\cos^2\theta)}{\{(1+e^2)^2\sin^2\theta+(1-e^2)^2\cos^2\theta\}^{\frac{1}{2}}}.$$

13. Show that
$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$
is the equation of the chord of the ellipse $x^2/a^2+y^2/b^2=1$, whose middle point is (x_1, y_1).

14. The locus of middle points of chords of the ellipse $x^2/a^2+y^2/b^2=1$, which subtend a right angle at its centre, is

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{a^2+b^2}{a^2b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2.$$

15. Show that if (x_1, y_1) is the middle point of a chord of the ellipse $x^2/a^2+y^2/b^2=1$, (ξ, η) the point of intersection of the normals and (x, y) that of the tangents at its extremities, then

$$\frac{a^2\xi}{x}+\frac{b^2\eta}{y}=(a^2-b^2)\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}\right).$$

16. If m_1, m_2 are the gradients of the tangents from the point (x, y) to the ellipse $x^2/a^2+y^2/b^2=1$, prove that they are the roots of the following quadratic equation in m:

$$(a^2-x^2)m^2+2xym+(b^2-y^2)=0.$$

17. Deduce from the result of Ex. 16 that if the tangents from the point (x, y) to the ellipse meet at an angle ϕ,

$$\tan\phi=\frac{2ab\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^{\frac{1}{2}}}{a^2+b^2-x^2-y^2}.$$

18. Prove that the locus of the point of intersection of rectangular tangents to the ellipse $x^2/a^2+y^2/b^2=1$ is the director-circle

$$x^2+y^2=a^2+b^2.$$

19. Show that the feet of the four normals from (x, y) to the ellipse $x^2/a^2+y^2/b^2=1$ are given by either of the equations

$$c^4\cos^4\theta-2c^2ax\cos^3\theta+(a^2x^2+b^2y^2-c^4)\cos^2\theta+2c^2ax\cos\theta-a^2x^2=0,$$

$$c^4\sin^4\theta+2c^2by\sin^3\theta+(a^2x^2+b^2y^2-c^4)\sin^2\theta-2c^2by\sin\theta-b^2y^2=0,$$

where $c^2=a^2-b^2$.

Prove that the coordinates of the centroid of the four feet are

$$(a^2x/2c^2,\ -b^2y/2c^2).$$

20. If $\theta_1, \theta_2, \theta_3, \theta_4$ are the eccentric angles of the feet of the normals from any point (x, y) to the ellipse $x^2/a^2+y^2/b^2=1$, prove that

$$\text{(i)}\ x=\frac{a^2-b^2}{2a}\Sigma\cos\theta; \quad \text{(ii)}\ y=\frac{b^2-a^2}{2b}\Sigma\sin\theta.$$

21. If the feet of two of the normals from a point coincide at the point θ, prove that the locus of the middle point of the join of the feet of the other two normals is

$$\left(\frac{xy}{ab}\right)^2=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^3.$$

22. Prove that if two lines drawn through the point $(\frac{3}{4}a, \frac{3}{4}b)$ meet the ellipse $x^2/a^2+y^2/b^2=1$ at four points, the normals at which are concurrent, one of the lines will be $4x/a-y/b=2$.

23. When two of the four normals to the ellipse $x^2/a^2+y^2/b^2=1$ coincide, prove that the line joining the feet of the other two is a normal of the ellipse

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=\frac{a^2b^2}{(a^2-b^2)^2}.$$

24. Find an equation whose roots are the gradients of the four normals that can be drawn from (x, y) to the ellipse $x^2/a^2+y^2/b^2=1$.

25. From any point (x', y') four normals are drawn to the ellipse $x^2/a^2+y^2/b^2=1$; prove that the tangents to the ellipse at the feet of these normals touch the parabola

$$(xx'-yy'-a^2+b^2)^2+4xx'yy'=0.$$

26. Prove that any tangent to the hyperbola

$$\frac{x^2}{a^3}-\frac{y^2}{b^3}=\frac{1}{a-b}$$

meets the conic $x^2/a^2+y^2/b^2=1$ in two points, the normals at which are equidistant from the centre.

27. If SY, $S'Y'$ are perpendiculars from the foci to the tangent at a point P on a hyperbola and NP is the ordinate to the transverse axis, prove that the angles SNY, SNY' are equal.

28. Show that the part of a common tangent of the curves

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{and} \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{a^2+b^2}{a^2-b^2}=0$$

intercepted between the points of contact subtends a right angle at the centre.

29. If the sum of the squares of the normals from a point to the curve $xy=a^2$ is constant, the point must lie on a circle.

30. Find the equation of the normal of the hyperbola $x^2/a^2-y^2/b^2=1$, drawn in a given direction, in the form

$$x\cos\alpha+y\sin\alpha=(a^2+b^2)\sin\alpha\cos\alpha(a^2\sin^2\alpha-b^2\cos^2\alpha)^{-\frac{1}{2}}.$$

31. From any point on the hyperbola $x^2/a^2-y^2/b^2=1$ straight lines are drawn perpendicular to the asymptotes and cutting the curve again in Q and Q'. Show that the envelope of QQ' is the hyperbola

$$\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2=\left(\frac{a^4+b^4}{a^4-b^4}\right)^2.$$

32. Show that the tangents to the rectangular hyperbola $x^2-y^2=a^2$ at the extremities of its latera recta pass through the vertices of the conjugate hyperbola $x^2-y^2=-a^2$.

33. If PN be the ordinate and PG the normal at a point P on a hyperbola, whose centre is C, and the tangent at P intersect the asymptotes at L and L', show that half the sum of CL and CL' is the mean proportional between CN and CG.

34. The tangents at the ends of a chord PQ of a hyperbola meet in T, and TM, TN are drawn parallel to the asymptotes to meet them in M, N. Prove that MN is parallel to PQ.

35. A variable tangent is drawn to the hyperbola $x^2-y^2=a^2$, cutting the circle $x^2+y^2=a^2$ in P and Q. Show that the locus of the middle point of PQ is the cardioid $(x^2+y^2)^2=a^2(x^2-y^2)$.

CHAPTER XXII.

POLE AND POLAR.

146. Joachimsthal's Section-Equation. Let T (Fig. 130) be the fixed point (x_1, y_1) and U the variable point (x, y), and let TU meet a conic in P_1, P_2; the study of the position-ratios of P_1, P_2 with respect to T and U, viz. TP_1/P_1U and TP_2/P_2U, as U varies under certain conditions, leads to important results. Let the conic be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad (1)$$

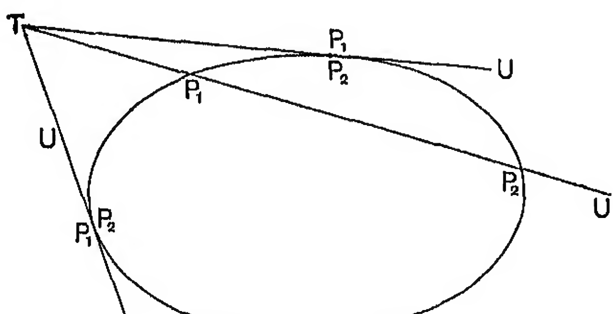


FIG. 130.

Let λ denote TP_1/P_1U (or TP_2/P_2U), then the coordinates of P_1 (or P_2) are

$$\frac{x_1+\lambda x}{1+\lambda}, \quad \frac{y_1+\lambda y}{1+\lambda}, \qquad (2)$$

Since P_1 (or P_2) lies on (1), the values given in (2) must satisfy (1); substitute these values in (1), then

$$\frac{(x_1+\lambda x)^2}{a^2}+\frac{(y_1+\lambda y)^2}{b^2}=(1+\lambda)^2$$

or

$$\lambda^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+2\lambda\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)+\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=0. \quad (3)$$

This is Joachimsthal's Equation. It is a quadratic in λ, whose roots are TP_1/P_1U and TP_2/P_2U. The student should work out the forms of the equation when the conics are $y^2=4ax$, $x^2/a^2-y^2/b^2=1$ and $xy=c^2$.

For the parabola $y^2=4ax$, Joachimsthal's Equation is

$$\lambda^2(y^2-4ax)+2\lambda\{yy_1-2a(x+x_1)\}+(y_1^2-4ax_1)=0. \quad (4)$$

For the hyperbola $x^2/a^2-y^2/b^2=1$,
Joachimsthal's Equation is

$$\lambda^2\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)+2\lambda\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)+\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)=0, \quad (5)$$

and for the hyperbola $xy=c^2$,

$$\lambda^2(xy-c^2)+2\lambda\left(\frac{xy_1}{2}+\frac{x_1y}{2}-c^2\right)+(x_1y_1-c^2)=0. \quad \ldots(6)$$

147. Pair of Tangents from a Point to a Conic. If U of the last section lie on either of the tangents from T (Fig. 130) to the conic, then $TP_1/P_1U=TP_2/P_2U$; the two roots of Joachimsthal's Equation are equal. Hence from (3) the pair of tangents from (x_1, y_1) to the ellipse $x^2/a^2+y^2/b^2=1$ is given by

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2;$$

from (5) the pair of tangents from (x_1, y_1) to the hyperbola $x^2/a^2-y^2/b^2=1$ is given by

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)^2;$$

from (4) the pair of tangents from (x_1, y_1) to the parabola $y^2=4ax$ is given by

$$(y^2-4ax)(y_1^2-4ax_1)=\{yy_1-2a(x+x_1)\}^2.$$

148. Pole and Polar. Definition. If a secant through a point T cut a conic in P_1 and P_2, and U be the harmonic conjugate of T with respect to P_1, P_2, the locus of U is the polar of T.

If (P_1P_2TU) in Fig. 131 is a harmonic range,

$$TP_1/P_1U = -TP_2/P_2U;$$

therefore the sum of the roots of Joachimsthal's Equation is zero. Hence the polar of (x_1, y_1) with respect to

(1) the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=1$,

(2) the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ is $\dfrac{xx_1}{a^2}-\dfrac{yy_1}{b^2}=1$,

(3) the parabola $y^2=4ax$ is $yy_1=2a(x+x_1)$.

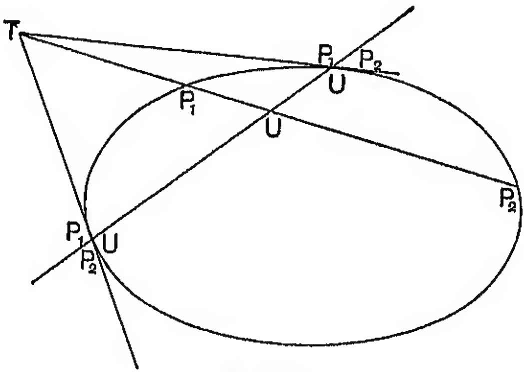


FIG. 131.

The polar of a point with respect to a conic is therefore a straight line, and the point is called the pole of the line.

The polar of a point outside a conic is the chord of contact of the pair of tangents from the point to the conic. As TP_1P_2 (Fig. 131) turns round T into the position of a tangent from T, $TP_1/P_1U = -TP_2/P_2U$, and T lies outside of P_1P_2, so that U lies between P_1 and P_2. When P_1 and P_2 run together at P, U also is at P, the point of contact;

hence the point of contact of each tangent from T lies on the polar of T. But the polar of T is a straight line, so that the chord of contact of the tangents is the polar.

149. Reciprocal Property of Pole and Polar. *If the point $A(x_1, y_1)$ lies on the polar of $B(x_2, y_2)$ with respect to a conic, the point B lies on the polar of A, and A and B are called conjugate points.* Let the conic be

$$x^2/a^2+y^2/b^2=1;$$

then the polar of $B(x_2, y_2)$ is

$$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1.$$

$A(x_1, y_1)$ lies on the polar of B; therefore

$$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}=1. \quad\text{.......................(1)}$$

Again the polar of $A(x_1, y_1)$ is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1; \quad\text{.......................(2)}$$

and x_2, y_2 satisfy equation (2), according to (1), so that B lies on (2), that is, B lies on the polar of A. A and B are called *conjugate points.* If two *lines* a, b are such that one passes through the pole of the other, it may be shown that the latter passes through the pole of the former, and the lines are called *conjugate lines.* If a pair of conjugate lines meet in T, then they are harmonically conjugate with respect to the tangents from T to the conic.

150. Examples of the Use of Pole and Polar. We shall now give some applications of the theory of pole and polar.

Ex. 1. **If a variable secant through a fixed point O, which lies outside or inside a conic, cut the conic in Q and Q′, and the tangents at Q and Q′ meet in T, the locus of T is the polar of O.**

QQ' (Fig. 132), the chord of contact of tangents from T, is the polar of T (§ 148), so that O lies on the polar of T; therefore, by the Reciprocal Property, T lies on the polar of O: in other words, the locus of T is the polar of O.

Ex. 2. **The polar of a point within a conic is parallel to the chord of the conic which is bisected at the point.**

Let V be the middle point of the chord QQ' in Fig. 132. Since QQ' is a chord through V, then, by definition, the harmonic conjugate of V with respect to Q and Q' lies on the polar of V; call the point I. Since V bisects QQ', I is the point at infinity on the line QQ' (§ 116). Again, T lies on the polar of V by Ex. 1; therefore IT is the polar of V. But IT is the parallel to QQ' through T, so that the polar of V is TK parallel to the chord bisected at V.

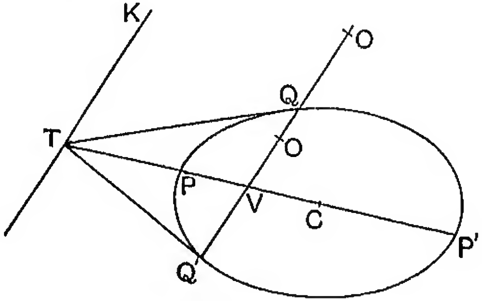


FIG. 132.

If V is (x_1, y_1) and the conic is $x^2/a^2+y^2/b^2=1$, the polar of V is $xx_1/a^2+yy_1/b^2=1$; hence the equation of QQ', the chord which is bisected at (x_1, y_1), is

$$(x-x_1)\frac{x_1}{a^2}+(y-y_1)\frac{y_1}{b^2}=0.$$

Further, all chords of the conic through C, the centre, are bisected at C; hence the chord through C which passes through I, the point at infinity on TK, QQ', is also bisected at C, so that the polar of I goes through C. But TV is the polar of I; therefore TV passes through C, and if it meet the conic in P, P' as in Fig. 132, $(PP'VT)$ is a harmonic range, since QQ' is the polar of T, and $CV \cdot CT = CP^2$.

Again, all chords parallel to QQ' pass through I, so that the polar of I is the locus of middle points of chords parallel to QQ', and the locus is therefore the straight line CV.

If V is the point (x_1, y_1) within the parabola $y^2=4ax$, the equation of QQ' is

$$(y-y_1)y_1=2a(x-x_1),$$

so that the gradient m of the chord whose middle point is (x_1, y_1) is $2a/y_1$, and therefore $y_1=2a/m$. Hence the middle points of chords of gradient m lie on the line $y=2a/m$, parallel to the axis; this line is the *diameter* for such chords. VQ is called the *ordinate* of Q with respect to the diameter PV. (See § 152.)

When V therefore lies within a *parabola* (Fig. 133), TV is parallel to the axis and $(TVP\infty)$ is a harmonic range, so that $TP=PV$. This

gives a simple construction for the polar of a point V within a parabola. *Draw VPT the diameter (parallel to the axis) through V to meet the parabola in P, and make PT equal to PV; the polar of V is the parallel TK through T to the tangent at P.*

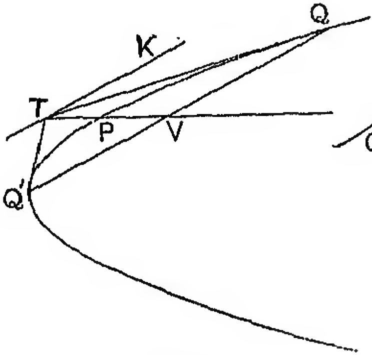


FIG. 133.

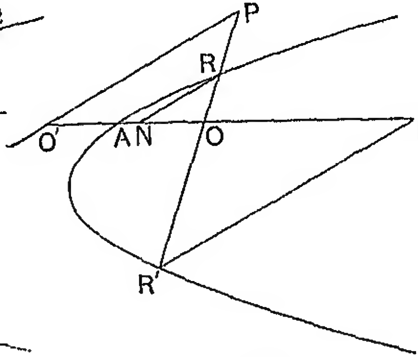


FIG. 134.

Ex 3. If NR, $N'R'$ are the ordinates of R, R' with respect to the diameter through a point A on a parabola which meets the chord RR' in O,

$$AO^2 = AN . AN'.$$

Produce OA (Fig. 134) to O' so that $OA = AO'$, then $O'P$ parallel to NR is the polar of O, as was seen in Ex. 2. Hence $(RR'OP)$ is a harmonic range (if RR' meet $O'P$ in P); therefore, by § 45, $(OO'NN')$ is a harmonic range, so that $AO^2 = AN . AN'$ (§ 44).

Ex. 4. If TQ, TQ' be tangents at Q and Q' on a parabola, the perpendicular from the focus S to QQ' bisects the intercept made by TQ, TQ' on the tangent at the vertex.

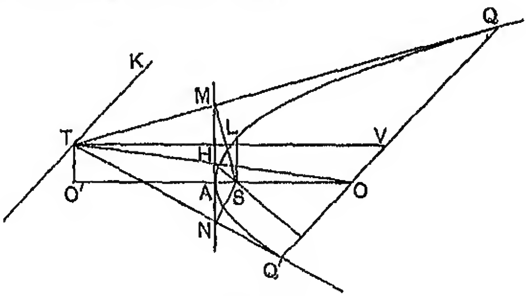


FIG. 135.

Let TQ, TQ' (Fig. 135) meet the tangent at the vertex in M, N, and

let V be the middle point of QQ'. Draw TK parallel to QQ'; then TK is the polar of V and $T(QQ'VK)$ is a harmonic pencil. Now SM, SN are perpendicular to TQ, TQ', and SL, the latus rectum, is perpendicular to TV. Therefore, if we draw SH perpendicular to TK or QQ' to meet MN in H, $S(MNLH)$ is a harmonic pencil; and MN is a transversal of this pencil parallel to the ray SL; therefore MN is bisected at H (§ 116).

It may be noted that if QQ' meets the axis in O (Fig. 135), TO also bisects MN. Draw TO' perpendicular to the axis to meet it in O'. TO' is the polar of O, so that $T(QQ'OO')$ is a harmonic pencil, of which the transversal MN is parallel to the ray TO', which shows that TO bisects MN. OT' is also bisected by MN; for the vertex A bisects OO', since TO' is the polar of O.

Ex. 5. The polar of the focus of a conic is the directrix, and the tangents at the ends of any focal chord cut the latus rectum produced in points equidistant from the focus.

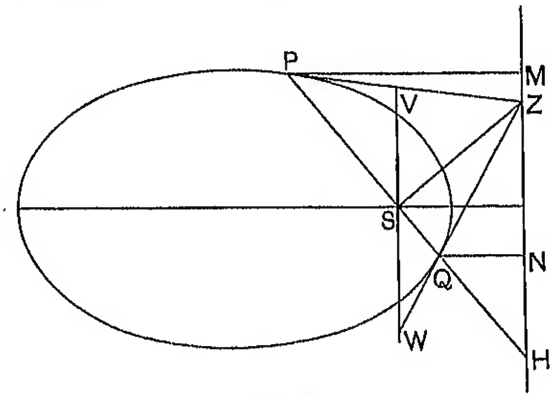


Fig. 136.

Let any focal chord PSQ (Fig. 136) meet the directrix in H and let M, N be the projections of P, Q on the directrix. Then

$$PS/SQ = MP/NQ = -PH/HQ,$$

so that PQ is cut harmonically at S and H. Hence the locus of H is the polar of S; in other words, the directrix is the polar of the focus.

If the tangents at P and Q meet the directrix in Z, then $Z(PQSH)$ is a harmonic pencil, and the latus rectum is a transversal parallel to the ray ZH; hence VW, the portion of it intercepted between ZP and ZQ, is bisected at S.

Ex. 6. PAB, PCD are secants of a conic ABCD drawn from a point P. If AC, BD meet in Q and AD, BC in R, show that QR is the polar of P.

Let X, Y in Fig. 137 be the harmonic conjugates of P with respect to A, B and C, D respectively.

Join QX, QY; then $Q(ABXP)$ and $Q(CDYP)$ are harmonic pencils. But QA, QB, QP are in line with QC, QD, QP. Therefore QX, QY are in one and the same straight line (§ 40); in other words, Q lies on XY. Now XY is the polar of P, therefore Q lies on the polar of P. Similarly R lies on the polar of P, so that QR is the polar of P.

Note. If QR meet the conic in T and T', we now know that PT, PT' are the tangents from P. The example shows how to draw the tangents from an external point to a conic by use of the ruler only.

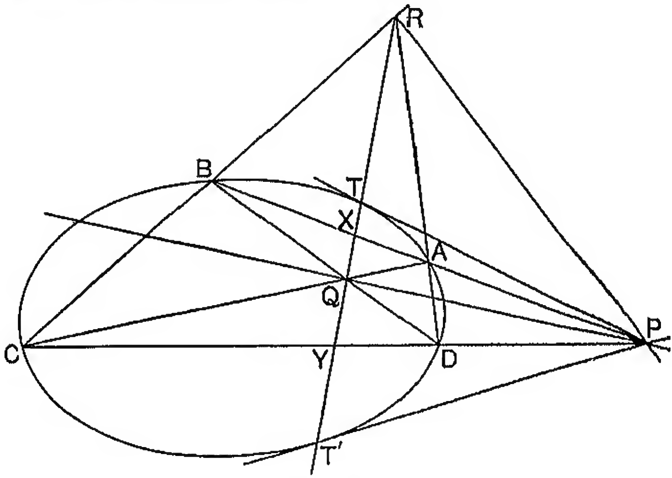


FIG. 137.

If A, B, C, D are any four points on a conic and AB and CD meet in P, AC and BD in Q, AD and BC in R, as in Fig. 137, we have seen by Ex. 7 that QR is the polar of P. Similarly PQ may be shown to be the polar of R, so that, by the Reciprocal Property, RP is the polar of Q. The triangle PQR is therefore such that each side is the polar of the opposite vertex; such a triangle is called a **self-conjugate triangle** or a **self-polar triangle**.

Ex. 7. The tangent at P on an ellipse cuts the auxiliary circle in Y and Y', and the other tangents from Y and Y' to the ellipse touch it at Q and Q'; show that QQ' meets the tangent YY' on the major axis and that YQ', $Y'Q$ intersect on the ordinate at P.

Let QY, $Q'Y'$ meet in O, and let QQ', YY' meet in T. Then QQ' is the polar of O with respect to the ellipse, so that T lies on the polar of O, and therefore O lies on the polar of T. But P, being the point of contact of the tangent TP, also lies on the polar of T; therefore OP is the polar of T. Hence if OP meet QQ' in M, $(QQ'MT)$ is a harmonic range; therefore $O(QQ'MT)$ is a harmonic pencil, and the range $(YY'PT)$ formed by the transversal PT is harmonic, by the fundamental theorem (§ 45). Now, if the tangent at P meet the major axis at T'', we have $T''Y/T''Y'=SY/S'Y'=PY/Y'P$, by similar triangles SPY, $S'PY'$. Hence T'' coincides with T, or QQ' and YY' meet on the major axis. Since T is on the major axis, its polar is perpendicular to the major axis; but OP is its polar, therefore OP is the ordinate at P. Now $(TPYY')$ and $(TMQQ')$ being harmonic ranges, it follows from § 46 that QY', $Q'Y$ cross on PM, that is, on the ordinate at P.

The following examples illustrate the use of pole and polar analytically.

Ex. 8. The locus of the poles of tangents to $x^2/a^2+y^2/b^2=1$ with respect to $x^2+y^2=a^2$ is the ellipse $a^2x^2+b^2y^2=a^4$.

Let (x_1, y_1) be a point on the locus; the polar of (x_1, y_1) with respect to $x^2+y^2=a^2$ is

$$xx_1+yy_1=a^2. \quad \text{.................................(i)}$$

If (i) touches $x^2/a^2+y^2/b^2=1$, then

$$a^2x_1^2+b^2y_1^2=a^4,$$

so that the locus is

$$a^2x^2+b^2y^2=a^4.$$

Ex. 9. The locus of poles of normal chords of the ellipse $x^2/a^2+y^2/b^2=1$ is the curve

$$\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2.$$

Let the equation of a normal chord be

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2, \quad \text{..............................(i)}$$

and let its pole be $(x_1 y_1)$; then (i) can be put in the form

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1. \quad \text{.................................(ii)}$$

From (i) and (ii), we have

$$\frac{a^3}{x_1\cos\theta}=-\frac{b^3}{y_1\sin\theta}=a^2-b^2$$

or

$$\frac{a^6}{x_1^2}=(a^2-b^2)^2\cos^2\theta \quad \text{and} \quad \frac{b^6}{y_1^2}=(a^2-b^2)^2\sin^2\theta;$$

hence, by addition,

$$\frac{a^6}{x_1^2}+\frac{b^6}{y_1^2}=(a^2-b^2)^2.$$

The locus of $(x_1 y_1)$ is therefore $a^6/x^2+b^6/y^2=(a^2-b^2)^2$.
The student should sketch the curve.

EXERCISES XLVIII.

1. If θ, ϕ are the eccentric angles of points P, Q on the ellipse $x^2/a^2+y^2/b^2=1$, prove that the coordinates of the pole of PQ are

$$a\cos\tfrac{1}{2}(\theta+\phi)/\cos\tfrac{1}{2}(\theta-\phi),\quad b\sin\tfrac{1}{2}(\theta+\phi)/\cos\tfrac{1}{2}(\theta-\phi).$$

2. Find the pole of the line $lx+my=1$ with respect to

(i) $x^2/a^2+y^2/b^2=1$; (ii) $x^2/a^2-y^2/b^2=1$;
(iii) $xy=c^2$; (iv) $y^2=4ax$.

3. Find the condition that $lx+my=1$, $l'x+m'y=1$ should be conjugate lines with respect to the conics (i)-(iv) in Ex. 2.

4. Find the equation of the chord of (i) $y^2=4ax$, (ii) $x^2/a^2-y^2/b^2=1$, which is bisected at the point $(x_1 y_1)$.

5. Two tangents are drawn from (α, β) to the ellipse $x^2/a^2+y^2/b^2=1$; show that the length of the chord of contact is

$$2ab(\alpha^2/a^4+\beta^2/b^4)^{\frac{1}{2}}.(\alpha^2/a^2+\beta^2/b^2-1)^{\frac{1}{2}}/(\alpha^2/a^2+\beta^2/b^2).$$

6. Prove that the polar with respect to a hyperbola of any point on an asymptote is parallel to that asymptote.

7. P and Q are two fixed points; through Q circles are drawn having a constant radius c, where $2c^2=PQ^2$; prove that the polars of P with respect to these circles touch a rectangular hyperbola whose centre is P.

8. Prove that the tangents at the extremities of all chords of the ellipse $x^2/a^2+y^2/b^2=1$ which subtend a right angle at the centre intersect on the ellipse $x^2/a^4+y^2/b^4=1/a^2+1/b^2$.

9. The polar of any point O with respect to a conic and the perpendicular to it from O meet either axis in T and G; prove that

$$CG.CT=CS^2.$$

10. R is the point θ on the ellipse $x^2/a^2+y^2/b^2=1$, RS and RS' meet the ellipse again in P and Q; prove that the coordinates of T, the pole of PQ, are

$$-a\cos\theta,\quad -\frac{b(1+e^2)}{1-e^2}\sin\theta.$$

11. The straight lines PS, PS' joining any point on the ellipse $x^2/a^2+y^2/b^2=1$ to the foci S, S' meet the curve again in Q, Q'. Tangents at Q, Q' meet in T. Show that the locus of T, as P moves round the curve, is the ellipse

$$(1+e^2)^2x^2/a^2+(1-e^2)^2y^2/b^2=(1+e^2)^2,$$

e being the eccentricity of the given ellipse.

12. Tangents drawn from any point on the parabola $y^2-2ax+c=0$ to touch the parabola $y^2=4ax$ meet the axis of x in the points E, F. Prove that E, F are equidistant from the pole of the common chord of the parabolas with respect to the parabola $y^2=4ax$.

13. If a circle touches a parabola at a given point, the pole of its chord of intersection with the parabola will lie on a fixed straight line.

14. Show that the locus of the feet of the perpendiculars let fall from points on a given diameter of a conic on the polar lines of those points is a rectangular hyperbola.

15. The pole of the normal at P to an ellipse is O and the foot of the perpendicular from the centre C on the tangent PO is Y; prove that the rectangle $YP \cdot PO$ is equal to the square on the semi-diameter conjugate to CP.

16. T is any point on the circle $x^2+y^2=a^2+b^2$, C is the centre of the ellipse $x^2/a^2+y^2/b^2=1$, TM and CN are perpendiculars to the polar of T with respect to the ellipse; prove that the rectangle $CN \cdot TM$ is constant.

17. S and S' are the foci of an ellipse, Q and Q' points on it on the same side of the major axis, such that SQ, $S'Q'$ are parallel and make an angle θ with the major axis. T is the pole of QQ', P is the point whose eccentric angle is θ and the tangent at P meets the major axis in T'. Show that TT' is at right angles to SQ.

18. Two points P and Q are such that the polar of one with respect to an ellipse passes through the other, and the line PQ passes through a fixed point; show that if P moves along a straight line through the centre of the ellipse, the locus of Q is a hyperbola.

19. A point P moves along the line $x+2y-3a=0$; show that its polar with respect to $y^2=4ax$ always passes through the point $(-3a, -4a)$.

20. If the polar of P with respect to the ellipse $x^2/a^2+y^2/b^2=1$ touches the ellipse $x^2/a'^2+y^2/b'^2=1$, prove that the locus of P is

$$a'^2x^2/a^4+b'^2y^2/b^4=1.$$

21. Prove that the pole of PQ with respect to a conic is the intersection of the polars of P and Q.

22. If two triangles ABC, $A'B'C'$ are such that the sides of $A'B'C'$ are the polars of A, B, C, prove that the sides of ABC are the polars of A', B', C'.

23. If any number of points are collinear, prove that their polars with respect to a conic are concurrent.

24. If tangents are drawn to a conic from points on a given straight line, the chords of contact pass through a fixed point.

25. Prove that the polar of a point P with respect to a conic centre C is parallel to the diameter of the conic whose direction is conjugate to that of CP.

26. If a, b, c, d are the polars of the points A, B, C, D, and if A, B, C, D form a harmonic range, prove that a, b, c, d form a harmonic pencil.

27. A conic touches the sides BC, CA, AB of a triangle at P, Q, R respectively, and QR meets BC at P'; show that $(BCPP')$ is a harmonic range, and that $\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = 1$.

28. AB, AC are the tangents from A to a conic. A variable tangent meets the conic at P and BC, CA, AB in Q, R, S respectively; prove that $(PQRS)$ is a harmonic range, and that BR, CS intersect on the line AP.

29. Parallel tangents to a conic at P, Q meet the tangent at the point R in S, T; PQ meets this tangent in U and PT, QS meet in V. Show that RV is the polar of U.

30. Two conics touch at a point P and intersect at points Q, R. Through P a line is drawn cutting the conics again in A and B; prove that the tangents at A and B intersect on QR.

31. Show that the polar, with respect to an ellipse, of any point on the auxiliary circle cuts the ellipse at the extremities of two parallel focal chords.

32. PSQ is a focal chord of a conic; PT is the tangent at P and the perpendicular through Q to PQ meets PT in T. Show that the directrix bisects QT.

33. If any line be drawn through a fixed point to cut a parabola, the tangents at the points of intersection will meet on a fixed straight line.

34. On a diameter of a parabola through the point P on the curve are cut off PA and PB so that P bisects AB. Show that the polar of A goes through B, and that the polars of A and B are parallel lines.

35. T is any point on the tangent at P on a parabola. A secant $TQOQ'$ meets the curve in Q, Q' and the diameter through P in O; show that
$$TO^2 = TQ \,.\, TQ'.$$

36. P is a point on a parabola, PV the diameter through P, V any point on the diameter, VM a perpendicular from V to the polar of V, meeting it at M. Show that the focus lies on MP.

37. H and K are points on the axis of a parabola equidistant from the vertex. Show that the segments of a chord through K, made by the axis, will subtend equal angles at H.

38. O is any point on the diameter of a parabola through a point P on the curve. Any line through O meets the curve in Q, Q' and the tangent at P in T. If T is the middle point of OR, prove that RQ, RO, RQ' are in harmonical progression.

39. TQ, TQ' are tangents at Q, Q' on a parabola whose focus is S, and QQ' cuts the axis in O. The diameter through T cuts the directrix in O'; show that TO and SO' bisect one another.

40. TQ, TQ' are tangents to a parabola and QQ' cuts the axis in O. If O' is the projection of T on the axis, prove that OO' is bisected at the vertex of the parabola.

41. TQ, TQ' are tangents to a parabola which meet the tangent at the vertex in M, N. If QQ' cuts the axis at O, show that the orthocentre of triangle MON is the focus.

42. The tangent at P on an ellipse meets the auxiliary circle in Y, Y', and YQ, $Y'Q'$ are the other tangents from Y, Y' to the ellipse. If QY, $Q'Y'$ meet in R and the tangents to the auxiliary circle at Y, Y' meet in R', prove that P, R, R' are collinear.

43. Tangents from a point P to the parabola $y^2-4ax=0$ are harmonic conjugates with respect to the tangents from P to the parabola $x^2+4by=0$; prove that the locus of P is the hyperbola $xy-2ab=0$.

44. Show that the locus of points from which the tangents to the ellipse $x^2/a^2+y^2/b^2=1$ and to its auxiliary circle form a harmonic pencil is a concentric ellipse, and find its equation.

45. Find the locus of the intersection of tangents to the conic $x^2/a^2+y^2/b^2=1$, which meet at an angle ϕ.

46. Find the equation of the locus of the intersection of perpendicular tangents to

$$\text{(i)}\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,; \qquad \text{(ii)}\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

CHAPTER XXIII.

DIAMETERS OF CONICS.

151. A Special Form of the Linear Equation. An important equation in the study of conics is

$$\frac{x-\xi}{\cos\theta}=\frac{y-\eta}{\sin\theta}=r \text{ ....(1)}$$

or
$$x=\xi+r\cos\theta,\quad y=\eta+r\sin\theta, \text{ ....(2)}$$

one form (§ 34) of the equation of the line through (ξ, η) of gradient $\tan\theta$, r measuring the step, positive or negative, from (ξ, η) to (x, y) along the line. The following examples will show the mode of its application.

Ex. 1. Let $A(\xi, \eta)$ be a specified point inside the parabola $y^2=4ax$. A chord PQ of the curve is bisected at A; find the equation of PQ.

The equations $x=\xi+r\cos\theta$, $y=\eta+r\sin\theta$ represent any line through (ξ, η); let them represent PQ. If (x, y) is the point P or Q, then $y^2=4ax$; therefore
$$(\eta+r\sin\theta)^2=4a(\xi+r\cos\theta)$$
or
$$r^2\sin^2\theta+2r(\eta\sin\theta-2a\cos\theta)+\eta^2-4a\xi=0, \text{ ....(3)}$$
a quadratic whose roots are AP and AQ. Now $AP+AQ=0$, hence
$$\eta\sin\theta-2a\cos\theta=0 \quad\text{or}\quad \tan\theta=2a/\eta.$$

Since PQ has gradient $2a/\eta$, its equation is $y-\eta=\dfrac{2a}{\eta}(x-\xi)$.

Ex. 2. Prove that the necessary and sufficient condition that (ξ, η) should lie within the parabola $y^2=4ax$ is that $\eta^2-4a\xi$ be negative.

Let A be the point (ξ, η); PQ a chord through A. Then, as in Ex. 1, AP, AQ are the roots of equation (3). But $AP \cdot AQ$, the product of the roots, is negative if and only if A lies within the curve; therefore the necessary and sufficient condition required is that $(\eta^2-4a\xi)/\sin^2\theta$ be negative, or that $\eta^2-4a\xi$ be negative.

Ex. 3. Find the equation of the tangent at (ξ, η) on the parabola $y^2=4ax$; find also the gradients and lengths of the tangents from (ξ, η) to the parabola, when (ξ, η) does not lie on the curve.

If (ξ, η) is on the curve, one root of equation (3) is zero; if $\tan\theta$ is the gradient of the tangent at (ξ, η), both roots are zero.

Hence the gradient of the tangent at (ξ, η) is given by

$$\eta\sin\theta-2a\cos\theta=0 \quad\text{or}\quad \tan\theta=2a/\eta,$$

so that the equation of the tangent is $y-\eta=\dfrac{2a}{\eta}(x-\xi)$ or $y\eta=2a(x+\xi)$, as in § 142.

Now suppose that (ξ, η) is not on the curve. If r in equation (3), Ex. 1, is the length of a tangent from (ξ, η) to the parabola, the roots of (3) are equal, whence

$$(\eta\sin\theta-2a\cos\theta)^2=\sin^2\theta(\eta^2-4a\xi) \quad\text{or}\quad \xi\tan^2\theta-\eta\tan\theta+a=0\,..(4)$$

The two values of $\tan\theta$ got by solving (4) are the gradients of the tangents from (ξ, η).

Also $r^2=$ product of roots of $(3)=(\eta^2-4a\xi)/\sin^2\theta$, where $\sin^2\theta$ is to be found from (4). We find that

$$4a^2r^2=(\eta^2-4a\xi)\{4a^2+[\eta\pm\sqrt{(\eta^2-4a\xi)}]^2\}.$$

Ex. 4. Through the point $A(14/5, 2/5)$ within the circle

$$5x^2+5y^2-14x-2y=40$$

are drawn the chords which are trisected at A; find the equations of the chords.

Let $x=\frac{14}{5}+r\cos\theta$, $y=\frac{2}{5}+r\sin\theta$ be the equations of PQ, a chord trisected at A. Substitute these values in the equation of the circle; the quadratic in r so obtained, namely

$$5r^2+2r(7\cos\theta+\sin\theta)-40=0, \quad\ldots\ldots\ldots\ldots(5)$$

has for roots AP, AQ; say r_1, r_2. Hence $r_1+2r_2=0$ or $2r_1+r_2=0$; therefore $(r_1+2r_2)(2r_1+r_2)=0$ or $2(r_1+r_2)^2+r_1r_2=0$, so that by (5)

$$(7\cos\theta+\sin\theta)^2=25, \text{ giving } \tan\theta=4/3 \text{ or } -3/4.$$

Hence the equations of the chords are $y-2/5=\frac{4}{3}(x-14/5)$ and $y-2/5=-\frac{3}{4}(x-14/5)$ or $4x-3y=10$ and $3x+4y=10$.

Ex. 5. The equation of the normal at the point θ on the ellipse $x^2/a^2+y^2/b^2=1$ may be put in the form

$$\frac{x-a\cos\theta}{b\cos\theta}=\frac{y-b\sin\theta}{a\sin\theta}=\frac{r}{CD}.$$

Let $\tan\phi$ be the gradient of the normal, then the equation of the normal may be written in the form

$$\frac{x-a\cos\theta}{\cos\phi}=\frac{y-b\sin\theta}{\sin\phi}=r. \quad\ldots\ldots\ldots\ldots(6)$$

But, by § 143, we have

$$\tan\phi=\frac{a\sin\theta}{b\cos\theta},$$

so that

$$\cos\phi=\pm\frac{b\cos\theta}{\sqrt{(a^2\sin^2\theta+b^2\cos^2\theta)}},\quad \sin\phi=\pm\frac{a\sin\theta}{\sqrt{(a^2\sin^2\theta+b^2\cos^2\theta)}}$$

or

$$\cos\phi=\pm\frac{b\cos\theta}{CD},\quad \sin\phi=\pm\frac{a\sin\theta}{CD}. \quad\text{.....................}(7)$$

If we select the two positive signs in equations (7), then r of equation (6) will be positive or negative according as $(x-a\cos\theta)$ and $\cos\theta$ have like or unlike signs; in other words, r will be positive when (x, y) lies on the *outward* normal and negative when (x, y) lies on the *inward* normal at θ.

By help of equation (7), equation (6) can now be written in the form

$$\frac{x-a\cos\theta}{b\cos\theta}=\frac{y-b\sin\theta}{a\sin\theta}=\frac{r}{CD}.$$

Ex. 6. On the normal at P to the ellipse $x^2/a^2+y^2/b^2=1$ are marked off PQ outwards and PQ' inwards, so that PQ and PQ' are each equal to the semi-diameter conjugate to CP; prove that

$$CQ=a+b \quad\text{and}\quad CQ'=a-b;$$

and deduce a construction for the axes of an ellipse when a pair of conjugate semi-diameters are given in magnitude and position.

Let P be the point $(a\cos\theta, b\sin\theta)$ and let CD be the semi-diameter conjugate to CP. Then the equation of the normal at P is, by Ex. 5,

$$\frac{x-a\cos\theta}{b\cos\theta}=\frac{y-b\sin\theta}{a\sin\theta}=\frac{r}{CD}.$$

If $r=CD$, $\quad x=(a+b)\cos\theta,\quad y=(a+b)\sin\theta$

and $\quad CQ^2=x^2+y^2=(a+b)^2 \quad\text{or}\quad CQ=a+b.$

If $r=-CD$, $\quad x=(a-b)\cos\theta,\quad y=-(a-b)\sin\theta$

and $\quad CQ'^2=x^2+y^2=(a-b)^2 \quad\text{or}\quad CQ'=a-b.$

Also $\quad$ the gradient of $CQ=\dfrac{(a+b)\sin\theta}{(a+b)\cos\theta}=\tan\theta$

and $\quad$ the gradient of $CQ'=-\dfrac{(a-b)\sin\theta}{(a-b)\cos\theta}=-\tan\theta.$

Hence the major axis bisects the angle between CQ and CQ'. Thus the directions of the axes are determined; their lengths are also determined, for $\quad CQ+CQ'=2a \quad\text{and}\quad CQ-CQ'=2b.$

When CP, CD are given in magnitude and position the points Q, Q' are found by drawing QPQ' perpendicular to CD and marking off PQ, PQ' equal to CD; the major axis is the bisector of the angle QCQ' and the lengths of the axes are given by the equations just written.

EXERCISES XLIX.

1. Find the equation of the chord of the ellipse $x^2/a^2+y^2/b^2=1$ which is bisected at the point (ξ, η).

2. Find the equation of the chord of the hyperbola $x^2/a^2-y^2/b^2=1$ which is bisected at the point (ξ, η).

3. The necessary and sufficient condition that the point (x_1, y_1) should lie within the ellipse $x^2/a^2+y^2/b^2=1$ is that $x_1{}^2/a^2+y_1{}^2/b^2-1$ be negative.

4. Use the method of § 151, Ex. 3, to find the equation of the tangent (i) at (x_1y_1) on $x^2/a^2+y^2/b^2=1$, (ii) at (x_1y_1) on $x^2/a^2-y^2/b^2=1$, (iii) at (x_1y_1) on $ax^2+2hxy+by^2+2gx+2fy+c=0$.

5. If TP, TQ are the tangents from $T(x, y)$ to the parabola $y^2=4ax$, whose focus is S, prove that

$$TP\,.\,TQ=(y^2-4ax)\,.\,\frac{TS}{a}.$$

6. A straight line through a fixed point $P(f, g)$ meets the lines $ax^2+2hxy+by^2=0$ at the points A, B and a point Q is taken on the line such that $1/PQ=1/PA+1/PB$ (PA, PB, PQ being steps); prove that the locus of Q is a straight line.

7. Find the length of the intercept made on the line $y=mx+c$ by the lines $ax^2+2hxy+by^2=0$.

8. Find the area of the triangle formed by the lines

$$lx+my=1, \quad ax^2+2hxy+by^2=0.$$

9. Q is a variable point on the line $ax+by+c=0$. On the line joining Q to the origin are marked off points P, P' such that $PQ=QP'=d$, a constant; find the locus of P, P'.

10. Prove that the locus of the points which divide in the ratio $1:k$ a series of chords inclined at an angle θ to the major axis of the conic $x^2/a^2+y^2/b^2=1$ is given by

$$4k\left(\frac{x\cos\theta}{a^2}+\frac{y\sin\theta}{b^2}\right)^2+(1-k)^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=0,$$

and draw some observations from this equation.

11. Prove that the square of the length of the chord of the ellipse $x^2/a^2+y^2/b^2=1$, which has its middle point at (h, k), is

$$4a^2b^2\left(\frac{h^2}{a^4}+\frac{k^2}{b^4}\right)\left\{\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^{-1}-1\right\}.$$

12. Find the equation of the chord of $2xy=c^2$ whose middle point is (α, β). Prove that the locus of the middle point of a chord of $2xy=c^2$, which is of constant length $2d$, is $c^2(x^2+y^2)=2xy(x^2+y^2-d^2)$.

13. On a chord of the parabola $y^2=4ax$ through a fixed point $O(h, k)$ a mean proportional OM is taken to the segments of the chord.

Show that the locus of M is the diameter whose equation is $y=k+c$, where c^2 is the numerical value of k^2-4ah.

14. Prove that the equation of the tangent to the ellipse $x^2/a^2+y^2/b^2=1$ at the point θ may be written in the form

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{r}{CD},$$

where r is the step (positive or negative) from the point of contact to (x, y).

15. If the tangent at P on an ellipse is met in T, T' by a pair of parallel tangents drawn at Q and Q', prove that $PT.PT'=CD^2$, and also that $QT.Q'T'$ is equal to the square of the semi-diameter parallel to QT.

16. If the tangent at P on an ellipse is met in L, L' by a pair of conjugate diameters, prove that $PL.PL'=CD^2$.

17. Find the equation of the normal at a point on an ellipse in the form

$$\frac{x-x_1}{\dfrac{px_1}{a^2}}=\frac{y-y_1}{\dfrac{py_1}{b^2}}=r,$$

where p is the perpendicular from the centre on the tangent at the point.

18. If the normal at P to the ellipse $x^2/a^2+y^2/b^2=1$ meets the axes in G, g and Q is a point on the normal such that

$$\frac{2}{PQ}=\frac{1}{PG}+\frac{1}{Pg},$$

show that the coordinates of Q are $\left(\dfrac{a^2-b^2}{a^2+b^2}a\cos\theta,\ \dfrac{b^2-a^2}{a^2+b^2}b\sin\theta\right)$.

152. The Parabola. The following are the leading theorems regarding the parabola.

THEOREM 1.

The locus of the middle points of parallel chords of the parabola

$$y^2=4ax$$

is the diameter

$$y=\frac{2a}{m},$$

where m is the common gradient of the chords.

Proof. Let $V(\xi, \eta)$ be the middle point of the chord QQ' (Fig. 138) of gradient m.

Let the equations of QQ' be

$$x=\xi+r\cos\theta, \quad y=\eta+r\sin\theta,$$

so that $\tan\theta=m$.

If x, y are the coordinates of Q or Q', then $y^2=4ax$, so that

$$r^2\sin^2\theta+2r(\eta\sin\theta-2a\cos\theta)+\eta^2-4a\xi=0. \quad \text{.......(1)}$$

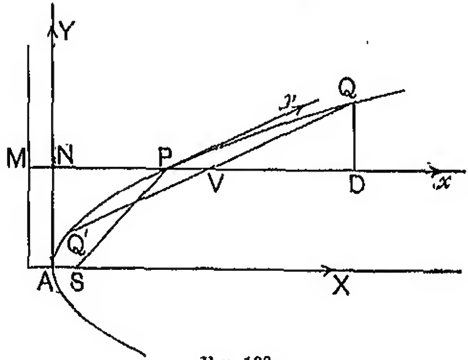


FIG. 138.

VQ, VQ' are the roots of (1), so that the sum of the roots is zero and therefore

$$\eta\sin\theta-2a\cos\theta=0 \quad \text{or} \quad \eta=\frac{2a}{m}. \quad \text{............(2)}$$

Writing y for η, we have

$$y=\frac{2a}{m}$$

as the locus of middle points of chords parallel to QQ'. The locus is a line parallel to the axis; let it meet the curve at P. PV is called a **diameter**, VQ and VQ' are **ordinates** of the diameter PV.

The tangent at P is the line obtained by moving QQ' parallel to itself till Q and Q' run together, so that the tangent at P is parallel to the chords bisected by the diameter through P. If NP, AN are the coordinates of P,

$$NP=\frac{a}{m^2} \quad \text{and} \quad AN=\frac{2a}{m}. \quad \text{................(3)}$$

Also $$SP=MP=a+\frac{a}{m^2}=a\operatorname{cosec}^2\theta. \quad \text{..............(4)}$$

THEOREM 2.

If QV is an ordinate of the diameter through a point P on a parabola, $QV^2=4SP.PV.$

Proof. (See Fig. 138.)

$$QV^2=-VQ.VQ'=\frac{4a\xi-\eta^2}{\sin^2\theta}, \text{ by (1).} \quad\text{.........(5)}$$

But, by (2) and (4),

$$\eta=\frac{2a}{m} \quad\text{and}\quad \frac{1}{\sin^2\theta}=\frac{SP}{a};$$

therefore, by (5),

$$QV^2=\frac{SP}{a}\left(4a\xi-\frac{4a^2}{m^2}\right)=4SP\left(\xi-\frac{a}{m^2}\right)$$

$$=4SP(NV-NP)=4SP.PV.$$

Cor. *If the diameter and tangent through P are oblique axes of x, y, then $VQ=y$ and $PV=x$, so that the equation of the parabola referred to these axes is*

$$y^2=4\alpha x, \quad\text{.........................(6)}$$

where $\alpha=SP$.

$4SP$ or 4α is called the parameter of the diameter through P.

The equation of the tangent at (x_1y_1) on (6) is

$$yy_1=2\alpha(x+x_1). \quad\text{.......................(7)}$$

Freedom equations for (6) are

$$x=\alpha t^2,\quad y=2\alpha t \quad\text{or}\quad x=\frac{\alpha}{m^2},\quad y=\frac{2\alpha}{m}; \quad\text{........(8)}$$

and the equation of the tangent at the point m is

$$y=mx+\frac{\alpha}{m}. \quad\text{..........................(9)}$$

If QD is the perpendicular from Q on PV, it is easy to show that

$$QD^2=4AS.PV. \quad\text{......................(10)}$$

153. Worked Examples. The following examples illustrate the theorems of the preceding section.

Ex. 1. The tangents to a parabola from a variable point T meet the tangent at the fixed point P in R and R', so that $PR \cdot PR'$ is constant; show that the locus of T is a straight line parallel to the tangent at P.

Refer the figure to the diameter and tangent through P as oblique axes of x and y, and let the equation of the curve be

$$y^2 = 4ax;$$

let MT, PM (Fig. 139) be the coordinates of T. Let TR, TR' meet the curve at Q, Q' and let Q, Q' be the points

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right), \quad \left(\frac{a}{m'^2}, \frac{2a}{m'}\right).$$

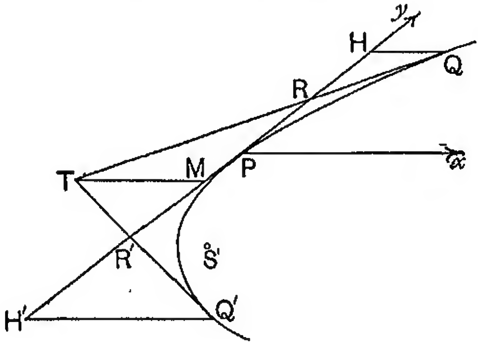


FIG. 139.

The equations of TR, TR' are

$$y = mx + \frac{a}{m} \quad \text{and} \quad y = m'x + \frac{a}{m'};$$

therefore $$MT = \frac{a}{mm'}, \quad PR = \frac{a}{m}, \quad PR' = \frac{a}{m'}.$$

Hence $$a \cdot MT = PR \cdot PR' = \text{constant},$$

so that MT is constant and the locus of T is a line parallel to the tangent at P.

Ex. 2. If a variable tangent to a parabola intersect three fixed tangents in Y_1, Y_2, Y_3, then the ratio $Y_1Y_2 : Y_2Y_3$ is constant.

Let the variable tangent touch the curve at P and refer the figure to the diameter and tangent through P as oblique axes of x and y. Let the equation of the parabola be

$$y^2 = 4ax,$$

and let the points of contact of the fixed tangents be

$$Q_1(x_1y_1), \quad Q_2(x_2y_2), \quad Q_3(x_3y_3).$$

The equation of the tangent at Q_1 is

$$yy_1=2a(x+x_1).$$

Put $x=0$, then $y=PY_1$, so that

$$PY_1=\frac{2ax_1}{y_1}=\frac{y_1}{2}.$$

Hence $$\frac{Y_1Y_2}{Y_2Y_3}=\frac{PY_2-PY_1}{PY_3-PY_2}=\frac{y_2-y_1}{y_3-y_2}=\text{a constant}.$$

Ex. 3. **Tangents at Q, Q′ on a parabola intersect in T, and the tangents at a third point P meets TQ, TQ′ in R, R′; show that**

$$\mathbf{QR : RT = TR' : R'Q' = RP : PR'}.$$

Using Fig. 139 and the notation of Ex. 1, we have

$$HQ=\frac{a}{m^2} \quad\text{and}\quad TM=\frac{a}{mm'};$$

therefore $$QR:RT=HQ:TM=\frac{m'}{m}.$$

Similarly $$TR':R'Q'=TM:H'Q'=\frac{a}{mm'}:\frac{a}{m'^2}=\frac{m'}{m}.$$

Again $$PR=\frac{a}{m},\quad PR'=\frac{a}{m'},$$

so that $$RP:PR'=\frac{m'}{m}.$$

154. Central Conics. We have already investigated the properties of diameters of the ellipse. The method employed does not apply to the hyperbola, so that we shall now give a short account of the general method of §§ 151, 152 as applied to the ellipse; certain obvious changes make the discussion apply to the hyperbola.

THEOREM 1.

If $\qquad y=mx \ldots\ldots(1) \quad$ *and* $\quad y=m'x \quad \ldots\ldots\ldots\ldots(2)$

are diameters of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

and if $$mm'=-\frac{b^2}{a^2} \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

each of the diameters (1) *and* (2) *bisects chords parallel to the other, and the diameters are called conjugate diameters.*

Let $V(\xi, \eta)$ be the middle point of QQ', a chord
se, of gradient m. The equations of QQ' may

$$x=\xi+r\cos\theta, \quad y=\eta+r\sin\theta,$$

$= m.$

e the coordinates of Q or Q', then $r=VQ$ or VQ'.

m (3), $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

$$\frac{(\xi+r\cos\theta)^2}{a^2}+\frac{(\eta+r\sin\theta)^2}{b^2}=1$$

$$+\frac{\sin^2\theta}{b^2}\Big)+2r\left(\frac{\xi\cos\theta}{a^2}+\frac{\eta\sin\theta}{b^2}\right)+\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}-1=0. \quad (5)$$

$+ VQ'=0$, so that the sum of the roots of (5)
refore

$$\frac{\cos\theta}{a^2}+\frac{\eta\sin\theta}{b^2}=0 \quad \text{or} \quad \eta=-\frac{b^2}{a^2}\cdot\frac{\cos\theta}{\sin\theta}\cdot\xi.$$

$\theta=m$ and, from (4), $m'=-b^2/a^2m$;

$$\eta=m'\xi,$$

e diameter (2) bisects all chords parallel to (1).
the diameter (1) bisects all chords parallel to (2).

THEOREM 2.

$$y=mx \quad \textit{and} \quad y=m'x$$

ters of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

$$mm'=\frac{b^2}{a^2},$$

e diameters bisects chords parallel to the other,
re called conjugate diameters.

THEOREM 3.

If P, D are the points

$$(a\cos\theta,\ b\sin\theta) \quad \textit{and} \quad (-a\sin\theta,\ b\cos\theta)$$

on the ellipse $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

CP, CD are conjugate semi-diameters and

$$CP^2+CD^2=a^2+b^2.$$

Proof. Let m, m' be the gradients of CP, CD; then

$$m=\frac{b\sin\theta}{a\cos\theta} \quad \text{and} \quad m'=-\frac{b\cos\theta}{a\sin\theta}.$$

Therefore $$mm'=-\frac{b^2}{a^2},$$

so that CP, CD are conjugate semi-diameters.

Also, $CP^2=a^2\cos^2\theta+b^2\sin^2\theta$, $CD^2=a^2\sin^2\theta+b^2\cos^2\theta$;

therefore $$CP^2+CD^2=a^2+b^2.$$

THEOREM 4.

If CP, CD are conjugate semi-diameters of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

and P is the point $(a\sec\theta, b\tan\theta)$, *then the coordinates of D may be put in the form*

$$(ai\tan\theta,\ bi\sec\theta),$$

where $i=\sqrt{-1}$; *and if* CD^2 *denote*

$$a^2\tan^2\theta+b^2\sec^2\theta,$$

then $$CP^2-CD^2=a^2-b^2.$$

Proof. Let m, m' be the gradients of CP, CD; then

$$m=\frac{b\tan\theta}{a\sec\theta} \quad \text{and} \quad mm'=\frac{b^2}{a^2}.$$

Therefore $$m'=\frac{b\sec\theta}{a\tan\theta}=\frac{bi\sec\theta}{ai\tan\theta}.$$

Now $(ai \tan\theta,\ bi \sec\theta)$ is a point on the hyperbola; therefore it is an extremity of the diameter $y=m'x$ conjugate to CP, and therefore it is D.

Also, $CP^2=a^2\sec^2\theta+b^2\tan^2\theta$, $CD^2=a^2\tan^2\theta+b^2\sec^2\theta$;

therefore $$CP^2-CD^2=a^2-b^2.$$

Theorem 5.

(1) *The equations of the tangent and normal at the point* $(a\cos\theta,\ b\sin\theta)$ *on the ellipse* $x^2/a^2+y^2/b^2=1$ *may be written in the forms*

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{r}{CD}$$

and $$\frac{x-a\cos\theta}{b\cos\theta}=\frac{y-b\sin\theta}{a\sin\theta}=\frac{r}{CD}.$$

(2) *The equations of the tangent and normal at the point* $(a\sec\theta,\ b\tan\theta)$ *on the hyperbola* $x^2/a^2-y^2/b^2=1$ *may be written in the forms*

$$\frac{x-a\sec\theta}{a\tan\theta}=\frac{y-b\tan\theta}{b\sec\theta}=\frac{r}{CD}$$

and $$\frac{x-a\sec\theta}{b\sec\theta}=\frac{y-b\tan\theta}{-a\tan\theta}=\frac{r}{CD}.$$

(See Ex. 5, p. 403.)

Theorem 6.

If PCP' is a diameter of a central conic which bisects the chord QQ' in V, then

$$\frac{QV^2}{PV.VP'}=\frac{CD^2}{CP^2},$$

where CD is the semi-diameter conjugate to CP.

Proof. Let V be the point $(\xi,\ \eta)$, let P be the point $(a\cos\theta,\ b\sin\theta)$, and let the equation of QQ' be

$$\frac{x-\xi}{-a\sin\theta}=\frac{y-\eta}{b\cos\theta}=\frac{r}{CD},$$

or $$x=\xi-\frac{a\sin\theta}{CD}\cdot r,\quad y=\eta+\frac{b\cos\theta}{CD}\cdot r.$$

If (x, y) are the coordinates of Q or Q', we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

so that $$\frac{r^2}{CD^2}-\frac{2r}{CD}\left(\frac{\xi\sin\theta}{a}-\frac{\eta\cos\theta}{b}\right)+\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}-1=0.$$

VQ, VQ' are the roots of this equation in r; therefore by the rule for the product of the roots of a quadratic equation, $$\frac{QV^2}{CD^2}=-\frac{VQ\,.\,VQ'}{CD^2}=-\left(\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}-1\right).\ \ldots\ldots\ldots\ldots(1)$$

The equation of $P'VP$ may be put in the form

$$\frac{x-\xi}{a\cos\theta}=\frac{y-\eta}{b\sin\theta}=\frac{r}{CP};$$

therefore, similarly,

$$\frac{VP\,.\,VP'}{CP^2}=\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}-1.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

But $VP\,.\,VP'=-PV\,.\,VP'$; therefore, from (1) and (2),

$$\frac{QV^2}{CD^2}=\frac{PV\,.\,VP'}{CP^2}\quad\text{or}\quad\frac{QV^2}{PV\,.\,VP'}=\frac{CD^2}{CP^2}.$$

THEOREM 7.

If a central conic be referred to CP and CD, a pair of conjugate semi-diameters, as oblique axes of x and y, its equation takes the form

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1\quad or\quad\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=1,$$

according as the conic is an ellipse or hyperbola, where $CP=\alpha$, $CD=\beta$.

Proof. If CV, VQ are the abscissa and ordinate of a point Q on the ellipse, then

$$QV^2=y^2,\quad PV\,.\,VP'=\alpha^2-x^2;$$

therefore, by Theorem 6,

$$\frac{y^2}{\alpha^2-x^2}=\frac{\beta^2}{\alpha^2} \quad \text{or} \quad \frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1.$$

The tangent at (x_1y_1) on the ellipse is

$$\frac{xx_1}{\alpha^2}+\frac{yy_1}{\beta^2}=1.$$

If $mm'=-\beta^2/\alpha^2$, $y=mx$ and $y=m'x$ are conjugate diameters of the ellipse.

155. Worked Examples. The following examples are applications of the theorems of the preceding section.

Ex. 1. **If parallel tangents at Q and Q′ on a central conic meet the tangent at P in T and T′, then**

$$PT \cdot PT'=CD^2 \quad \text{and} \quad QT \cdot Q'T'=CE^2,$$

where CD and CE are the semi-diameters parallel to the tangents at P and Q respectively.

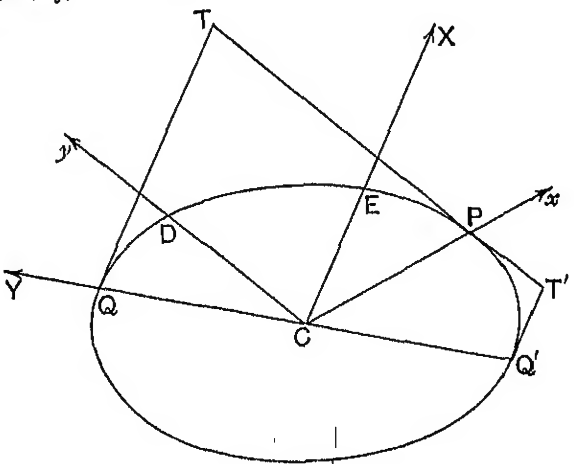


Fig. 140.

First, refer the figure to CP, CD as oblique axes of x, y (Fig. 140); let the equation of the conic be

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1,$$

and let the coordinates of Q, Q' be (x_1, y_1), $(-x_1, -y_1)$.

The equations of QT, $Q'T'$ are

$$\frac{xx_1}{\alpha^2}+\frac{yy_1}{\beta^2}=1 \quad \text{and} \quad \frac{xx_1}{\alpha^2}+\frac{yy_1}{\beta^2}=-1.$$

The common abscissa of T, T' is α. Put $x=\alpha$ in these equations; we get

$$\frac{yy_1}{\beta^2}=1-\frac{x_1}{\alpha} \quad \text{and} \quad \frac{yy_1}{\beta^2}=-1-\frac{x_1}{\alpha}.$$

Therefore, since $y=PT$ in the first of these and $y=PT'$ in the second, we have after multiplication,

$$PT\,.\,PT'\,.\frac{y_1^2}{\beta^4}=-\left(1-\frac{x_1^2}{\alpha^2}\right)=-\frac{y_1^2}{\beta^2},$$

and therefore, dropping consideration of the sign,

$$PT\,.\,PT'=\beta^2=CD^2.$$

Second, refer the figure to CE, CQ as oblique axes of x, y; let the equation of the conic be

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1,$$

where $\alpha=CE$, $\beta=CQ$ in this case. Let P be the point (x_1y_1).
Then the equation of TPT' is

$$\frac{xx_1}{\alpha^2}+\frac{yy_1}{\beta^2}=1.$$

The ordinate of T is β. Put $y=\beta$; we get

$$\frac{xx_1}{\alpha^2}=1-\frac{y_1}{\beta} \quad \text{or} \quad QT\,.\frac{x_1}{\alpha^2}=1-\frac{y_1}{\beta}.$$

The ordinate of T' is $-\beta$. Put $y=-\beta$ in the equation of TPT'; we get

$$\frac{xx_1}{\alpha^2}=1+\frac{y_1}{\beta} \quad \text{or} \quad Q'T'\,.\frac{x_1}{\alpha^2}=1+\frac{y_1}{\beta}.$$

Hence $$QT\,.\,Q'T'\,.\frac{x_1^2}{\alpha^4}=1-\frac{y_1^2}{\beta^2}=\frac{x_1^2}{\alpha^2},$$

so that $$QT\,.\,Q'T'=\alpha^2=CE^2.$$

Ex. 2. If the chord PR of the conic $x^2/a^2-y^2/b^2=1$ and the tangent at P are equally inclined to the axes and PR meet the axes at Q and q, then

$$QP\,.\,Pq=CD^2.$$

Let P be the point $(a\sec\theta,\ b\tan\theta)$ and let the equation of PR be put in the form

$$\frac{x-a\sec\theta}{a\tan\theta}=\frac{y-b\tan\theta}{-b\sec\theta}=\frac{r}{CD}$$

according to Theorem 5.
When $y=0$, $r=PQ$; when $x=0$, $r=Pq$.

Therefore $$PQ=CD\,.\frac{\tan\theta}{\sec\theta} \quad \text{and} \quad Pq=-CD\,.\frac{\sec\theta}{\tan\theta},$$

and therefore $$PQ\,.\,Pq=-CD^2 \quad \text{or} \quad QP\,.\,Pq=CD^2.$$

Ex. 3. Tangents from a variable point T meet a conic in Q and Q', and the tangent at the fixed point P on the conic meets TQ and TQ' in R and R'; if $PR \cdot PR'$ is constant, the locus of T is a straight line.

Refer the figure to CP and the conjugate semi-diameter as oblique axes of x and y; let the equation of the conic be

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1,$$

and let the coordinates of T be $(x_1 y_1)$.

The equation of the tangent-pair TQ, TQ' is

$$\left(\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}-1\right)\left(\frac{x_1{}^2}{\alpha^2}+\frac{y_1{}^2}{\beta^2}-1\right)=\left(\frac{xx_1}{\alpha^2}+\frac{yy_1}{\beta^2}-1\right)^2.$$

Put $x=\alpha$; PR and PR' are the roots of the resulting quadratic in y.

Hence $$PR \cdot PR' = -\frac{(x_1-\alpha)}{(x_1+\alpha)} \cdot \beta^2.$$

Since $PR \cdot PR'$ is constant, x_1 is constant, so that the locus of T is a straight line parallel to the tangent at P.

156. Asymptotes. Similar treatment may be applied to asymptotes.

Put $\qquad x=\xi+r\cos\theta \quad$ and $\quad y=\eta+r\sin\theta$

in the equations

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \;\ldots\ldots(1) \quad \text{and} \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=0. \;\ldots\ldots\ldots\ldots(2)$$

We then get the quadratics

$$r^2\left(\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}\right)+2r\left(\frac{\xi\cos\theta}{a^2}-\frac{\eta\sin\theta}{b^2}\right)+\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-1=0 \quad (3)$$

and

$$r^2\left(\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}\right)+2r\left(\frac{\xi\cos\theta}{a^2}-\frac{\eta\sin\theta}{b^2}\right)+\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}=0. \quad (4)$$

If (ξ, η) is the middle point V of a chord QQ' of (1) and if QQ' meet the asymptotes (2) in R and R', then

$$VQ+VQ'=0,$$

so that, by (3), $$\frac{\xi\cos\theta}{a^2}-\frac{\eta\sin\theta}{b^2}=0,$$

and therefore, by (4), $\quad VR+VR'=0,$

so that we have

THEOREM 1.

If the diameter CPV of a hyperbola meet the curve in P and bisect the chord QQ' in V, V is also the middle point of RR', the intercept made on QQ' by the asymptotes.

If QQ' move parallel to itself till V coincides with P then we have, as in § 132, Theorem 6,

THEOREM 2.

The part of the tangent at P intercepted between the asymptotes is bisected at P.

THEOREM 3.

If QQ', a chord of a hyperbola, meet the asymptotes in R and R', and the tangent at P parallel to QQ' meet an asymptote in T,

$$RQ \,.\, RQ' = RQ \,.\, QR' = PT^2 = CD^2,$$

where CP, CD are conjugate semi-diameters.

Proof. Let R be the point (ξ, η) of equation (3); then $\dfrac{\xi^2}{a^2} - \dfrac{\eta^2}{b^2} = 0$ by (2), and

$$RQ \,.\, RQ' = -\frac{1}{\dfrac{\cos^2\theta}{a^2} - \dfrac{\sin^2\theta}{b^2}}.$$

Now $\tan\theta$ is the gradient of QQ' or CD; but the gradient of CD, by Theorem 4 of § 154, is $b\sec\phi / a\tan\phi$, where P is the point ϕ, so that

$$\frac{\cos^2\theta}{a^2} = \frac{\tan^2\phi}{CD^2} \quad\text{and}\quad \frac{\sin^2\theta}{b^2} = \frac{\sec^2\phi}{CD^2}.$$

Hence $$RQ \,.\, RQ' = CD^2.$$

Since V is the middle point of both QQ' and RR' (Th. 1)

$$RQ \,.\, QR' = RQ \,.\, RQ' = CD^2.$$

When R coincides with T, $RQ \,.\, RQ'$ becomes TP^2. Hence the theorem is established.

If QQ' is perpendicular to the transverse axis,

$$RQ \cdot RQ' = RQ \cdot QR' = CB^2;$$

if it is perpendicular to the conjugate axis,

$$RQ \cdot RQ' = RQ \cdot QR' = CA^2.$$

EXERCISES L.

1. Prove that the intercepts PG and Pg, made on the normal at P between P and the axes, are

$$\frac{b}{a} CD \quad \text{and} \quad \frac{a}{b} CD.$$

2. If θ is the eccentric angle of P, a point on an ellipse, prove that

$$\tan^2\theta = (CD^2 - b^2)/(a^2 - CD^2).$$

3. If α is the angle which the tangent at P makes with the focal distance of P, prove that $\sin\alpha = b/CD$.

4. Prove that $\cos CPG = ab/CP \cdot CD$.

5. If the diameter conjugate to CP meet the normal at P in F, prove that $PF \cdot PG = BC^2$ and $PF \cdot Pg = CA^2$, where G, g are the intersections of the normal and the axes.

6. If CP, CD are conjugate semi-diameters of an ellipse, if CD meets SP at E and PL is the projection of PG on SP, prove that (1) $PE = CA$, (2) angle PEg is a right angle, (3) $PL = CB^2/CA$.

7. If the tangent at P on a hyperbola meet the asymptotes in L, L', prove that $PL = PL' = CD$.

8. If the tangent at P on a hyperbola is cut by any pair of parallel tangents in T', T'', prove that $PT' \cdot PT'' = CD^2$.

9. If the tangent at P on a hyperbola is cut by any pair of conjugate diameters in T', T'', prove that $PT' \cdot PT'' = CD^2$.

10. The asymptotes of a conic are harmonically conjugate with respect to any pair of conjugate diameters.

11. If $y^2 = 4\alpha x$ is the equation of a parabola referred to oblique axes of x, y inclined at an angle ω, prove that the equation of the directrix referred to the same oblique axes is

$$x + y\cos\omega + \alpha = 0.$$

12. The ordinates through a point P on a parabola of the diameters LV, $L'V'$ through the extremities L, L' of the latus rectum are PV, PV'; show that $VV'^2 = 4LL' \cdot SP$.

13. Prove that, if P and P' are any two points on a parabola, the mean proportional between the distances of P and S from the tangent at P' is half the distance between the diameters through P and P'.

14. If the tangent at Q on a parabola meet the diameter PV in T, prove that $TQ^2=4TP.SQ$.

15. Using the equation $QD^2=4AS.PV$ for a parabola (§ 152), prove that

$$(ax+by+c)^2=k(px+qy+r)$$

is the equation of a parabola, $ax+by+c=0$ being the equation of a diameter and $px+qy+r=0$ that of the tangent at its vertex. Find the latus rectum of the parabola.

16. If O is the middle point of the chord QQ' of a hyperbola which meets the asymptotes in R, R' and cuts any pair of conjugate diameters in K, K', prove that $OR^2=OK.OK'$.

17. Prove that conjugate diameters of a rectangular hyperbola are equally inclined to each of the asymptotes.

18. Two tangents to an ellipse are drawn parallel to the normal at a point P on the ellipse, and the normal is equally inclined to the axes. Prove that the semi-diameter parallel to those tangents is a mean proportional between the perpendiculars from P on the two tangents.

19. Find the greatest value of the angle between a diameter of an ellipse of eccentricity e and the normal at its extremity; and show that for the earth's orbit round the sun, of which the eccentricity may be taken as 1/60, this angle is less than half a minute of arc.

20. Prove that, if PG, the normal at P to a parabola, cut the axis in G, the length of the chord drawn through G parallel to the tangent at P varies as the focal distance of P.

21. PR is a chord of a parabola such that PR and the tangent at P form an isosceles triangle with the axis; prove that PR is divided in the ratio 1 : 3 by the axis.

CHAPTER XXIV.

NERAL THEOREMS ON CONICS. CONFOCAL CONICS. CURVATURE.

General Equation of the Second Degree. Before g to certain general theorems regarding a conic, we ind a general form for the equation of a conic, so ave needless repetitions of proofs. Let $(x_1 y_1)$ be the and $x\cos\alpha+y\sin\alpha-p=0$, the directrix of a conic eccentricity is e; then the equation of the conic is

$$(x-x_1)^2+(y-y_1)^2=e^2(x\cos\alpha+y\sin\alpha-p)^2$$

$$.\ -e^2\cos^2\alpha)x^2-2e^2\sin\alpha\cos\alpha\, xy+y^2(1-e^2\sin^2\alpha)$$

$$(x_1-e^2p\cos\alpha)-2y(y_1-e^2p\sin\alpha)+x_1{}^2+y_1{}^2-e^2p^2=0.$$

general equation of the second degree, namely

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

l by $f(x, y)=0$ or $S=0$, *represents any conic.* If >0, it represents a *hyperbola*; for $ax^2+2hxy+by^2$ n two real and distinct factors, and the curve goes ity in two different real directions. If $h^2-ab=0$, esents a conic going to infinity in but one real n (or in two coincident real directions), that is, a *la*. If $h^2-ab<0$, it represents a conic which does to infinity in any real direction, that is, an *ellipse*. e already shown in Chapter XII. how to draw rough f the graphs of $f(x, y)=0$.

If $f(x, y)=0$ is a central conic, show that the coordinates of its e the solutions of $ax+hy+g=0$, $hx+by+f=0$.

$_1$, y_1) be the centre of $f(x, y)=0$, and shift the origin to the y putting $x=\xi+x_1$, $y=\eta+y_1$. The equation of the conic

$$h\xi\eta+b\eta^2+2\xi(ax_1+hy_1+g)+2\eta(hx_1+by_1+f)+f(x_1, y_1)=0. \quad (1)$$

Think of (ξ, η) and $(-\xi, -\eta)$ as the two extremities of a diameter of the conic; then it is clear that $(-\xi, -\eta)$ must satisfy (1), so that

$$a\xi^2+2h\xi\eta+b\eta^2-2\xi(ax_1+hy_1+g)-2\eta(hx_1+by_1+f)+f(x_1, y_1)=0. \quad (2)$$

Subtracting (2) from (1) and dividing by 4, we get

$$\xi(ax_1+hy_1+g)+\eta(hx_1+by_1+f)=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Now (3) is true for all values of ξ, η belonging to the conic; therefore

$$ax_1+hy_1+g=0 \quad \text{and} \quad hx_1+by_1+f=0.$$

Ex. 2. **The equation of a central conic referred to the centre as origin is found by writing ξ for x, η for y in the terms of the second degree and $x_1/2$, $y_1/2$ for x, y in the terms of the first degree, where (x_1, y_1) is the centre.**

Equation (1) of Ex. 1 gives the equation of the conic referred to its centre as origin. Since $ax_1+hy_1+g=0$ and $hx_1+by_1+f=0$, it remains to prove that $f(x_1, y_1)=2g(x_1/2)+2f(y_1/2)+c$.

Now $x_1(ax_1+hy_1+g)=0$ and $y_1(hx_1+by_1+f)=0$; by addition we see that $ax_1^2+2hx_1y_1+by_1^2+gx_1+fy_1=0$.

To each add $\quad 2g(x_1/2)+2f(y_1/2)+c$;

then $\quad f(x_1, y_1)=2g(x_1/2)+2f(y_1/2)+c.$

Ex. 3. **Find the latus rectum and the equations of the axis and tangent at the vertex of the parabola**

$$x^2-2xy+y^2-6x+2y+9=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

We write (i) in the form

$$(x-y)^2=6x-2y-9,$$

then in the form

$$(x-y+\lambda)^2=2x(3+\lambda)-2y(1+\lambda)+\lambda^2-9. \quad \ldots\ldots\ldots\ldots(ii)$$

Next, we determine λ so that the lines

$$x-y+\lambda=0 \quad \text{and} \quad 2x(3+\lambda)-2y(1+\lambda)+\lambda^2-9=0$$

may be perpendicular. This gives $\dfrac{3+\lambda}{1+\lambda}=-1$ or $\lambda=-2$.

Going back to (ii), we substitute $\lambda=-2$, and so get (i) into the form

$$(x-y-2)^2=2x+2y-5. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(iii)$$

To compare (iii) with $NP^2=4AS \, . \, AN$ (§ 125),

we write it $\quad \dfrac{(x-y-2)^2}{2}=\sqrt{2} \, . \, \dfrac{2x+2y-5}{2\sqrt{2}}.$

Hence the latus rectum $=\sqrt{2}$; the equation of the axis is $x-y-2=0$ and of the tangent at the vertex $2x+2y-5=0$. The rough form of this parabola is shown in Fig. 84, p. 220.

Ex. 4. **Find the lengths and the equations of the axes of the ellipse**

$$5x^2-4xy+4y^2-20x+8y-44=0. \quad\text{.......................(i)}$$

Here $a=5$, $h=-2$, $b=4$, $g=-10$, $f=4$, $c=-44$.

We first find the centre according to Ex. 1:

$$ax_1+hy_1+g=0 \quad\text{and}\quad hx_1+by_1+f=0 \quad\text{give}\quad x_1=2,\ y_1=0.$$

Applying Ex. 2, we find for the equation of the ellipse referred to its centre as origin,
$$5\xi^2-4\xi\eta+4\eta^2=64. \quad\text{..........................(ii)}$$

The equation of a concentric circle, radius r, is
$$\xi^2+\eta^2=r^2. \quad\text{...............................(iii)}$$

Multiplying (ii) by r^2 and (iii) by 64 and subtracting, we obtain
$$\xi^2(5r^2-64)-4r^2\xi\eta+\eta^2(4r^2-64)=0. \quad\text{..................(iv)}$$

Equation (iv) gives two straight lines through the common centre and the intersections of (ii) and (iii). (Compare § 125, Ex. 3.) When r^2 is equal to the square of either semi-axis of (i), these two lines coincide, and then
$$4r^4=(5r^2-64)(4r^2-64) \quad\text{or}\quad r^4-36r^2+256=0,$$
giving
$$r^2=18+2\sqrt{17} \quad\text{or}\quad 18-2\sqrt{17}. \quad\text{....................(v)}$$

These are the squares of the semi-axes.

When the two lines (iv) coincide, (iv) may be written
$$[\xi(5r^2-64)-2r^2\eta]^2=0,$$
so that

when $r^2=18+2\sqrt{17}$, $\xi(5r^2-64)-2r^2\eta=0$ is the major axis;

when $r^2=18-2\sqrt{17}$, $\xi(5r^2-64)-2r^2\eta=0$ is the minor axis.

Hence the major and minor axes are
$$\xi(13+5\sqrt{17})=2\eta(9+\sqrt{17}) \quad\text{and}\quad \xi(13-5\sqrt{17})=2\eta(9-\sqrt{17})$$
or, referred to the x and y axes,
$$(x-2)(13+5\sqrt{17})=2y(9+\sqrt{17}) \quad\text{and}\quad (x-2)(13-5\sqrt{17})=2y(9-\sqrt{17}).$$

The rough form of the ellipse is shown in Fig. 86, p. 223.

Note. The same method applies to the hyperbola; one of the values of r^2 in the equation corresponding to (v) will be positive and one negative; the first is CA^2, the second $-CB^2$. The student may apply the method to the hyperbolas of §§ 91, 93.

Ex. 5. The area of the ellipse whose equation is
$$ax^2+2hxy+by^2=1$$
is $\pi/\sqrt{ab-h^2}$.

Rotate the axes through an angle θ and put
$$x=\xi\cos\theta-\eta\sin\theta \quad\text{and}\quad y=\xi\sin\theta+\eta\cos\theta.$$

The equation of the ellipse referred to the ξ and η axes becomes

$$a'\xi^2+2h'\xi\eta+b'\eta^2=1, \qquad \text{(i)}$$

where
$$a'=a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta,$$
$$h'=(b-a)\sin\theta\cos\theta+h(\cos^2\theta-\sin^2\theta),$$
$$b'=a\sin^2\theta-2h\sin\theta\cos\theta+b\cos^2\theta.$$

We get (i) into the canonical form of the equation of the ellipse if we choose $h'=0$,

or $\quad (b-a)\sin\theta\cos\theta+h(\cos^2\theta-\sin^2\theta)=0 \quad$ or $\quad \tan 2\theta=\dfrac{2h}{a-b}$.

Equation (i) then becomes

$$a'\xi^2+b'\eta^2=1. \qquad \text{(ii)}$$

The semi-axes of the ellipse are therefore $\dfrac{1}{\sqrt{a'}}$, $\dfrac{1}{\sqrt{b'}}$, and the area is therefore

$$\frac{\pi}{\sqrt{(a'b')}}.$$

Now (p. 102, Ex. 20) $a'b'=a'b'-h'^2=ab-h^2$, as is easily verified. Therefore the area of the ellipse is $\pi/\sqrt{(ab-h^2)}$.

Ex. 6. **Prove that the equation of a conic referred to the tangent and normal at a point as axes of x and y may be written in the form**

$$y=ax^2+2hxy+by^2.$$

Also show that, if through a given point on a conic two lines at right angles to each other be drawn to meet the curve, the line joining their extremities will pass through a fixed point on the normal.

Let $\quad Ax^2+2Hxy+By^2+2Gx+2Fy+C=0$

be the equation of the conic.

The conic passes through the origin; therefore $C=0$.

And the tangent at the origin is $y=0$; therefore $x=0$ *twice* gives the roots of $Ax^2+2Gx=0$, so that $G=0$.

Hence the equation of the conic takes the form

$$Ax^2+2Hxy+By^2+2Fy=0$$

or
$$y=(-A/2F)x^2+2(-H/2F)xy+(-B/2F)y^2,$$

which is of the given form.

In the second part of the example, refer the figure to the tangent and normal at the point as axes of x and y; let the equation of the conic be

$$y=ax^2+2hxy+by^2,$$

and let the equation of the line joining the extremities of the perpendicular chords through the origin be

$$lx+my=1.$$

Then
$$y(lx+my)=ax^2+2hxy+by^2$$

is the equation of the pair of perpendicular lines through the origin. Since they are perpendicular

$$a+b-m=0,$$

so that $lx+my=1$ may be written in the form

$$lx+(a+b)y=1,$$

which always passes through the intersection of $x=0$ and $(a+b)y=1$, that is, through a fixed point on the normal.

Note. If $a+b=0$, the conic is a rectangular hyperbola, and $m=0$. Hence the particular case: If through any point on a rectangular hyperbola be drawn two chords at right angles, the perpendicular let fall from the point on the line joining their extremities is the tangent to the curve at the point.

158. Forms of the Conic. We have taken the general equation of the second degree, viz.

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{............(1)}$$

to represent a conic. It follows that the term conic therefore includes a pair of straight lines (which may be parallel or coincident) as well as a circle, parabola, ellipse or hyperbola. And this is not surprising, for if the hyperbola $x^2/a^2-y^2/b^2=1-\lambda$ is traced for varying values of λ between 0 and 1, it will differ very little from its asymptotes when λ is all but 1, so that the hyperbola may be imagined to be "squashed" into a pair of straight lines which are its asymptotes; the hyperbola is said to *degenerate* into these lines. Similarly, a parabola may degenerate into a pair of parallel straight lines. To see the significance of the term conic if *defined* by equation (1), we must *start* from (1) and investigate its different forms. We shall give only the outline of the investigation.

If $h^2 \neq ab$, we can shift the origin to the point (x_1, y_1) by putting $x=\xi+x_1$, $y=\eta+y_1$, where $ax_1+hy_1+g=0$ and $hx_1+by_1+f=0$, and (1) will then take the form

$$a\xi^2+2h\xi\eta+b\eta^2=\frac{abc+2fgh-af^2-bg^2-ch^2}{h^2-ab}. \quad \text{.....(2)}$$

The expression $abc+2fgh-af^2-bg^2-ch^2$ is called the **discriminant** of equation (1); we shall denote it by D.

If $h^2 \neq ab$ and $D=0$, we see from (2) that (1) will represent *two intersecting straight lines.*

If $h^2 \neq ab$ and $D \neq 0$, equation (2) may be put in the form

$$A\xi^2 + 2H\xi\eta + B\eta^2 = 1. \qquad \text{.................(3)}$$

We may, by rotation from ξ, η axes to u, v axes, as in §157, Ex. 5, bring (2) or (3) to the form

$$\alpha u^2 + \beta v^2 = \gamma. \qquad \text{.......................(4)}$$

If $\gamma = 0$ and α, β have the same sign, then $u = 0$ and $v = 0$, so that (1) would represent a *point*, for example one of the limiting points of a system of coaxal circles.

If $\gamma \neq 0$, we see from (4) that (1) may be brought to the canonical form of the equation of the *circle, ellipse* or *hyperbola*.

So far we have supposed that $h^2 - ab \neq 0$. If $h^2 - ab = 0$, then (1) will either take the form, as in §157, Ex. 3,

$$(\alpha x + \beta y + \gamma)^2 = k(\beta x - \alpha y + \delta), \qquad \text{...........(5)}$$

so that it represents a *parabola*, or the form

$$(\alpha x + \beta y + \gamma)^2 = e, \qquad \text{.....................(6)}$$

so that it represents a *pair of parallel lines*.

159. General Theorems. We shall now give the main theorems regarding the general conic whose equation is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \qquad \text{........(1)}$$

THEOREM I.

If from the point $P(x_1, y_1)$ a line of gradient $\tan\theta$ *be drawn to meet the conic (1) in Q and R, then PQ and PR are the roots of the following quadratic in r:*

$$\begin{aligned} &r^2(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta) \\ &\quad + 2r\{(ax_1 + hy_1 + g)\cos\theta + (hx_1 + by_1 + f)\sin\theta\} \\ &\quad + ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0. \qquad \text{........(2)} \end{aligned}$$

This equation is obtained by putting $x = x_1 + r\cos\theta$, $y = y_1 + r\sin\theta$ in (1).

THEOREM 2.

The equation of the tangent at the point (x_1, y_1) *on the conic* (1) *is*

$$axx_1+h(xy_1+x_1y)+byy_1+g(x+x_1)+f(y+y_1)+c=0. \quad (3)$$

By expressing the condition that both roots of (2) are zero, we get the value of $\tan\theta$, the gradient of the tangent at (x_1, y_1), to be

$$-\frac{ax_1+hy_1+g}{hx_1+by_1+f},$$

and the equation of the tangent is then $y-y_1=(x-x_1)\tan\theta$, which may be put in the form (3).

THEOREM 3.

The locus of the middle points of chords of the conic (1), *which have a common gradient* m, *is the straight line whose equation is*

$$ax+hy+g+m(hx+by+f)=0. \quad \text{..............}(4)$$

If $m=\tan\theta$ and (x_1, y_1) is a point on the locus, then the sum of the roots of (2) is zero, so that

$$(ax_1+hy_1+g)\cos\theta+(hx_1+by_1+f)\sin\theta=0$$

or

$$ax_1+hy_1+g+m(hx_1+by_1+f)=0;$$

hence (4) is the equation of the locus.

THEOREM 4.

If m, m' *are the gradients of a pair of conjugate diameters of the conic* (1), *then*

$$a+h(m+m')+bmm'=0. \quad \text{.................}(5)$$

For, according to (4), $m'=-(a+mh)/(h+mb)$.

THEOREM 5.

The chord of the conic (1), *whose middle point is* (x_1, y_1), *is given by the equation*

$$(x-x_1)(ax_1+hy_1+g)+(y-y_1)(hx_1+by_1+f)=0. \quad \text{...}(6)$$

Put

$$\tan\theta=(y-y_1)/(x-x_1)$$

in the investigation of Theorem 3.

THEOREM 6.

If the line joining the point $T(x_1, y_1)$ *to the point* $U(x, y)$ *meet the conic* (1) *in* P_1, P_2, *then the values of the ratios* TP_1/P_1U, TP_2/P_2U *are the roots of the following quadratic in* λ:

$$\lambda^2(ax^2+2hxy+by^2+2gx+2fy+c)$$
$$+2\lambda\{axx_1+h(xy_1+x_1y)+byy_1+g(x+x_1)+f(y+y_1)+c\}$$
$$+(ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c)=0. \quad \ldots\ldots\ldots\ldots(7)$$

This is **Joachimsthal's Equation**, and is found by proceeding exactly as in § 146.

THEOREM 7.

The equation of the pair of tangents from the point (x_1, y_1) *to the conic* (1) *is*

$$(ax^2+2hxy+by^2+2gx+2fy+c)(ax_1^2+2hx_1y_1+2gx_1$$
$$+2fy_1+c)$$
$$=\{axx_1+h(xy_1+x_1y)+byy_1+g(x+x_1)+f(y+y_1)+c\}^2. \; (8)$$

Proceed exactly as in § 147.

Note. The pair of tangents from a focus of a conic satisfy the circular conditions, namely, the term in xy is absent and the coefficients of x^2, y^2 are equal.

THEOREM 8.

The polar of (x_1, y_1) *with respect to the conic* (1) *is given by the equation*

$$axx_1+h(xy_1+x_1y)+byy_1+g(x+x_1)+f(y+y_1)+c=0. \; (9)$$

Proceed as in § 148.

THEOREM 9.

The asymptotes of the conic (1) *are given by the equation*

$$ax^2+2hxy+by^2+2gx+2fy+c$$
$$+\frac{abc+2fgh-af^2-bg^2-ch^2}{h^2-ab}=0. \; \ldots(10)$$

If c' be added to both sides of (1), we get

$$ax^2+2hxy+by^2+2gx+2fy+c+c'=c'.$$

The left-hand expression is the product of two linear factors if

$$ab(c+c')+2fgh-af^2-bg^2-(c+c')h^2=0,$$

that is, if $$c'=\frac{abc+2fgh-af^2-bg^2-ch^2}{h^2-ab}.$$

But each of the factors equated to zero gives an asymptote, hence the product of the factors equated to zero gives both asymptotes in one equation, and it is equation (10).

160. Confocal Conics. A system of central conics having their foci in common is called a confocal system.

The general equation of a system of conics confocal with the ellipse $x^2/a^2+y^2/b^2=1$ is easily seen to be

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1, \quad \text{.....................(1)}$$

where λ is a variable parameter; because

$$(a^2+\lambda)-(b^2+\lambda)=a^2-b^2=a^2e^2,$$

so that the foci remain fixed as λ varies.

The following theorems are of interest.

Theorem 1.

Through every point in the plane of the ellipse $x^2/a^2+y^2/b^2=1$ two confocal conics can be drawn, one an ellipse and the other a hyperbola.

Let (1) be the equation of a confocal through the point (x_1, y_1); then to determine λ, we have the quadratic

$$(\lambda+a^2)(\lambda+b^2)-(\lambda+a^2)y_1^2-(\lambda+b^2)x_1^2=0. \quad \text{......(2)}$$

Now the graph of the left-hand expression is a parabola whose concavity is upwards, and which crosses the λ-axis since the value of the expression is $(a^2-b^2)x_1^2$, a positive quantity, when $\lambda=-a^2$, and $(b^2-a^2)y_1^2$, a negative quantity, when $\lambda=-b^2$. Hence the roots of (2) are real, so that two real confocals pass through (x_1, y_1). Further, if λ_1 and λ_2

are the roots of (2), both $a^2+\lambda_1$ and $a^2+\lambda_2$ are positive, but $b^2+\lambda_1$ and $b^2+\lambda_2$ have opposite signs, so that one confocal is an ellipse and the other a hyperbola.

THEOREM 2.

The two conics confocal with the ellipse $x^2/a^2+y^2/b^2=1$ and passing through any point (x_1, y_1) cut at right angles.

For if λ_1, λ_2 specify the two confocals, we have

$$\frac{x_1^2}{a^2+\lambda_1}+\frac{y_1^2}{b^2+\lambda_1}=1=\frac{x_1^2}{a^2+\lambda_2}+\frac{y_1^2}{b^2+\lambda_2}$$

or $$(\lambda_1-\lambda_2)\left\{\frac{x_1^2}{(a^2+\lambda_1)(a^2+\lambda_2)}+\frac{y_1^2}{(b^2+\lambda_1)(b^2+\lambda_2)}\right\}=0.$$

Since $\lambda_1-\lambda_2 \neq 0$, we get the condition that the tangents at (x_1, y_1) on the confocals, namely

$$\frac{xx_1}{a^2+\lambda_1}+\frac{yy_1}{b^2+\lambda_1}=1 \quad \text{and} \quad \frac{xx_1}{a^2+\lambda_2}+\frac{yy_1}{b^2+\lambda_2}=1,$$

are perpendicular.

Ex. 1. Only one of the conics of the system confocal with

$$x^2/a^2+y^2/b^2=1$$

can be drawn to touch a given line.

Ex. 2. Variable parallel tangents to $x^2/a^2+y^2/b^2=1$ meet a common perpendicular in Q and R, which again meets a parallel tangent to a confocal in P; prove that $PQ \cdot PR$ is constant.

Ex. 3. Prove that the locus of the points of contact of tangents from a fixed point to a system of confocal conics is a cubic curve which passes through the point and the foci of the system.

Ex. 4. Prove that $y^2=4\lambda(x+\lambda)$, where λ is a variable parameter, represents a system of confocal parabolas.

161. Freedom Equations of Conics. Let

$$x=\frac{at^2+bt+c}{lt^2+mt+n}, \quad y=\frac{a't^2+b't+c'}{lt^2+mt+n} \quad \dots\dots\dots\dots(1)$$

be freedom equations of a curve, then the curve is a conic; because the curve given by these equations meets any line $Ax+By+C=0$ in two points. (See IV., p. 290.)

If lt^2+mt+n has real and distinct factors, the conic goes to infinity in two real and distinct directions, so that (1) represents a *hyperbola* if $m^2-4ln>0$.

If lt^2+mt+n has a repeated factor $(t-\alpha)^2$, in other words if lt^2+mt+n is a perfect square, the conic goes to infinity in one direction, so that (1) represents a *parabola* if $m^2-4ln=0$. In this case, by putting t for $1/(t-\alpha)$, we get the form used in Ex. 2 below.

If lt^2+mt+n has imaginary factors, the conic is a closed curve, so that (1) represents an *ellipse* if $m^2-4ln<0$.

Ex. 1. Find the asymptotes of the conic whose freedom equations are

$$x=\frac{2t^2-t+1}{(t-1)(t-2)}, \quad y=\frac{t^2+3t+2}{(t-1)(t-2)}. \quad \text{..................(i)}$$

Let $lx+my=1$ be an asymptote; then $lx+my=1$ touches (i) where x, y are infinite, that is where $t=1$ or $t=2$.

Hence $$l(2t^2-t+1)+m(t^2+3t+2)=(t-1)(t-2)$$

or $$t^2(2l+m-1)-t(l-3m-3)+(l+2m-2)=0 \quad \text{............(ii)}$$

is a quadratic in t, whose roots are $t=1$ twice or $t=2$ twice.

If the roots of (ii) are $t=1$ twice,

$$2l+m-1=l+2m-2 \quad \text{and} \quad l-3m-3=2(2l+m-1),$$

so that $l=-3/4$, $m=1/4$ and the asymptote is $3x-y+4=0$.

If the roots of (ii) are $t=2$ twice,

$$l-3m-3=l+2m-2 \quad \text{and} \quad l-3m-3=4(2l+m-1),$$

so that $l=12/35$, $m=-1/5$ and the asymptote is $12x-7y=35$.

Ex. 2. Prove that the equations

$$x=at^2+bt+c, \quad y=a't^2+b't+c'$$

represent a parabola whose latus rectum is

$$\frac{(ab'-a'b)^2}{(a^2+a'^2)^{\frac{3}{2}}}.$$

Solving the given equations for t^2, t, we find

$$t^2=\frac{b'x-by+bc'-b'c}{ab'-a'b}, \quad t=\frac{a'c-ac'-a'x+ay}{ab'-a'b},$$

so that the constraint equation of the locus of (x, y) is

$$(a'x-ay+ac'-a'c)^2=(ab'-a'b)(b'x-by+bc'-b'c), \quad \text{........(i)}$$

and this represents a parabola; $a'x-ay+ac'-a'c=0$ is a diameter of the curve and $b'x-by+bc'-b'c=0$ is the tangent at the extremity of the diameter.

Comparing equation (i) with the equation

$$QD^2 = 4AS \cdot PV$$

of § 152, we see that the latus rectum $4AS$ is

$$\frac{(ab'-a'b)}{a^2+a'^2}\sqrt{b^2+b'^2} \cdot \sin\theta,$$

where θ is the angle between the diameter and the tangent.

Now

$$\tan\theta = \frac{ab'-a'b}{ab+a'b'}, \text{ and therefore } \sin\theta = \frac{ab'-a'b}{\sqrt{\{(a^2+a'^2)(b^2+b'^2)\}}},$$

so that the latus rectum is $\dfrac{(ab'-a'b)^2}{(a^2+a'^2)^{\frac{3}{2}}}$.

Ex. 3. Prove that

$$x = a\cos\theta + b\sin\theta + c, \quad y = a'\cos\theta + b'\sin\theta + c'$$

are freedom equations of an ellipse.

Solve for $\cos\theta$ and $\sin\theta$, then square and add. Or put $2t/(1+t^2)$ for $\sin\theta$ and $(1-t^2)/(1+t^2)$ for $\cos\theta$.

Ex. 4. Draw the conics specified by the following freedom equations, and state the nature of each:

(i) $x = \dfrac{2t}{t^2-1}$, $y = \dfrac{1+t^2}{1-t^2}$; (ii) $x = 2t^2 - 5t + 2$, $y = 3t^2 + t - 2$;

(iii) $x = \dfrac{2t^2-9t+9}{(t-1)^2}$, $y = \dfrac{t(5-2t)}{(t-1)^2}$; (iv) $x = \dfrac{t^2-t+1}{t^2+t+1}$, $y = \dfrac{t^2-3t+2}{t^2+t+1}$.

Ex. 5. Prove that the gradient of the tangent at the point t on the parabola

$$x = at^2 + 2bt + c, \quad y = a't^2 + 2b't + c'$$

is $(a't+b')/(at+b)$, and find the value of t for the vertex.

EXERCISES LI.

1. Trace the conics specified by the following equations, giving the axes and their equations when the conic is an ellipse or hyperbola, and the latus rectum and the equations of the axis and the tangent at the vertex when the conic is a parabola.

(i) $2x^2 + y^2 = 2xy + 2y$; (ii) $6x^2 - xy - y^2 - x + 3y + 2 = 0$;

(iii) $3x^2 - 2xy + 3y^2 = 32$;

(iv) $13x^2 + 28xy - 8y^2 - 10x - 20y + 61 = 0$;

(v) $7x^2 - 48xy - 7y^2 + 110x - 20y + 100 = 0$;

(vi) $9x^2 + 6xy + y^2 + 2x + 3y + 4 = 0$;

(vii) $5x^2 - 4xy + 8y^2 - 6x - 12y - 36 = 0$;

(viii) $9x^2 + 16y^2 - 24xy - 50x - 100y + 225 = 0$;

(ix) $y^2 - 4xy - 5x^2 + 6y + 42x - 63 = 0$.

2. Trace the conic

$$\left(\frac{x}{a}+\frac{y}{b}-1\right)^2=\left(\frac{x}{a}-1\right)\left(\frac{y}{b}-1\right),$$

showing its relationship to the lines $x=a$, $y=b$, $x/a+y/b=1$.

3. Prove that the conics

$$x^2-y^2-4x+2y+2=0, \quad x^2+3y^2-4x-6y+4=0$$

cut orthogonally at each of the four points of intersection.

4. Find the condition that the lines $a'x^2+2h'xy+b'y^2=0$ may be conjugate diameters of the conic $ax^2+2hxy+by^2=1$.

Prove that the equation of the equi-conjugate diameters is

$$(a^2-ab+2h^2)x^2+2h(a+b)xy+(b^2-ab+2h^2)y^2=0.$$

5. Find the directrix and coordinates of the focus of the conic

$$x^2+2xy+y^2-3x+6y-4=0.$$

6. Prove that the directrix of the parabola

$$x=at^2+2bt+c, \quad y=a't^2+2b't+c'$$

is

$$ax+a'y=ac+a'c'-b^2-b'^2.$$

7. Find the latus rectum, the equation of the directrix and the equation of the tangent at the point t on the parabola

$$x=t^2\sin\alpha+2at+b, \quad y=t^2\cos\alpha+2a't+b'.$$

8. If the equation of a conic is

$$ax^2+2hxy+by^2=1,$$

prove (1) that the squares of the semi-axes are given by the following equation:

$$\frac{1}{r^4}-\frac{1}{r^2}(a+b)+ab-h^2=0\,;$$

and (2) that the positions of the axes are given by the equation

$$\left(a-\frac{1}{r^2}\right)x+hy=0.$$

9. If the line-pair $ax^2+2hxy+by^2=0$ represents the asymptotes of a hyperbola, find the equation of the pair of axes.

10. Find the equation of the conic passing through the origin and having the same asymptotes as the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

11. Prove that the latus rectum of the parabola

$$(\alpha x+\beta y)^2+2gx+2fy+c=0$$

is

$$2(\alpha f-\beta g)/(\alpha^2+\beta^2)^{\frac{3}{2}}.$$

Also prove that the equation of the axis is $\alpha x+\beta y+\lambda=0$, where $\lambda=(\alpha g+\beta f)/(\alpha^2+\beta^2)$.

12. Prove that the gradient at the point (x, y) on the parabola

$$(\alpha x+\beta y)^2+2gx+2fy+c=0$$

is
$$-(\alpha^2 x+\alpha\beta y+g)/(\alpha\beta x+\beta^2 y+f),$$

and that the equation of the axis may be formed by equating this expression to β/α.

13. If m is the gradient at the point P on the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

whose centre is C, prove that the equation of CP is

$$ax+hy+g+m(hx+by+f)=0.$$

Deduce that the axes are given by the equation

$$hX^2-(a-b)XY-hY^2=0,$$

where $X\equiv ax+hy+g$, $Y\equiv hx+by+f$.

14. If a conic is given by the general equation of the second degree, the eccentricity e is given by the equation

$$e^4+\frac{(a-b)^2+4h^2}{ab-h^2}(e^2-1)=0.$$

15. From the fact (§ 159, Theorem 7, *Note*) that the tangents to a conic from a focus satisfy the conditions for a circle, prove that if $f(x, y)=0$ is the general equation of a conic, the foci are the points given by

$$\frac{X^2-Y^2}{a-b}=\frac{XY}{h}=f(x, y),$$

where $X\equiv ax+hy+g$, $Y\equiv hx+by+f$.

162. The Rectangle Theorem. The Rectangle Theorem is a general theorem for the conic which has many applications.

If two variable secants of a conic whose directions are fixed cut the conic in P, Q and P', Q' and intersect in O, then

$$\frac{OP\,.\,OQ}{OP'\,.\,OQ'}$$

is constant for all positions of O.

We shall use the general equation $f(x, y)=0$ or

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad\qquad (1)$$

for the equation of the conic.

Let (ξ, η) be the coordinates of O and let $x=\xi+r\cos\theta$, $y=\eta+r\sin\theta$ be the equations of OPQ. Substitute these

values for x, y in (1) and arrange the result as a quadratic in r, thus:

$$r^2(a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta)$$
$$+2r\{\cos\theta(a\xi+h\eta+g)+\sin\theta(h\xi+b\eta+f)\}+f(\xi,\eta)=0. \quad (2)$$

OP, OQ are the roots of this equation, so that

$$OP\,.\,OQ=\frac{f(\xi,\eta)}{a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta}. \quad \text{.........(3)}$$

Now let $x=\xi+r'\cos\theta'$, $y=\eta+r'\sin\theta'$ be the equations of $OP'Q'$; then

$$OP'\,.\,OQ'=\frac{f(\xi,\eta)}{a\cos^2\theta'+2h\sin\theta'\cos\theta'+b\sin^2\theta'}. \quad \text{......(4)}$$

From (3) and (4) we get

$$\frac{OP\,.\,OQ}{OP'\,.\,OQ'}=\frac{a\cos^2\theta'+2h\sin\theta'\cos\theta'+b\sin^2\theta'}{a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta}. \quad \text{......(5)}$$

Now the directions of OPQ, $OP'Q'$ are fixed so that $\tan\theta$, $\tan\theta'$ being the gradients of OPQ, $OP'Q'$ are fixed; hence the right-hand side of (5) is constant, so that $OP\,.\,OQ/OP'\,.\,OQ'$ is also constant.

163. Examples on the Rectangle Theorem. We shall now work some examples on this theorem.

Ex. 1. **If VQ is an ordinate of the diameter POP′ of a conic and if CP, CD are conjugate semi-diameters, then**

$$\frac{QV^2}{PV\,.\,VP'}=\frac{CD^2}{CP^2}.$$

Let QV, CD (Fig. 110) meet the conic again in Q', D'; then DCD', QVQ' have the same direction, and PCP', PVP' are in the same direction. Therefore

$$\frac{VQ\,.\,VQ'}{VP\,.\,VP'}=\frac{CD\,.\,CD'}{CP\,.\,CP'} \quad\text{or}\quad \frac{QV^2}{PV\,.\,VP'}=\pm\frac{CD^2}{CP^2},$$

the upper sign holding for an ellipse and the under sign for a hyperbola according to our use of CD^2 for the hyperbola (§ 154). These equations are the equivalents of $x^2/\alpha^2+y^2/\beta^2=1$, $x^2/\alpha^2-y^2/\beta^2=1$ of § 154, Theorem 7.

Ex. 2. **Let PQ, P'Q' be two chords of a conic intersecting at O; let op and op', CD and CD', LSK and L'SK' be pairs of tangents, semi-diameters, and focal chords parallel to OPQ, OP'Q' respectively; then**

$$\frac{\mathrm{OP}\,.\,\mathrm{OQ}}{\mathrm{OP'}\,.\,\mathrm{OQ'}}=\frac{\mathrm{op}^2}{\mathrm{op'}^2}=\frac{\mathrm{CD}^2}{\mathrm{CD'}^2}=\frac{\mathrm{SL}\,.\,\mathrm{SK}}{\mathrm{SL'}\,.\,\mathrm{SK'}}=\frac{\mathrm{LK}}{\mathrm{L'K'}}.$$

Let OPQ, $OP'Q'$ (Fig. 141) move, keeping their directions fixed, till P, Q coincide at p, and P', Q' coincide at p', then O will coincide with o; $OP\,.\,OQ$ becomes op^2 and $OP'\,.\,OQ'$ becomes op'^2. Hence by the Rectangle Theorem $OP\,.\,OQ/OP'\,.\,OQ'=op^2/op'^2$.

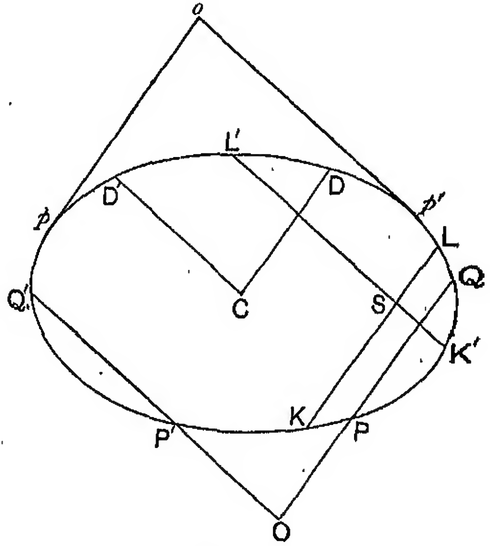


Fig. 141.

Now let PQ move so as to keep its direction fixed till it coincides with the CD-line, and let $P'Q'$ move into coincidence with the CD'-line; then O becomes C, so that $OP\,.\,OQ/OP'\,.\,OQ'=CD^2/CD'^2$.

It will be clear that by moving O into the position of the focus S, $OP\,.\,OQ/OP'\,.\,OQ'=SL\,.\,SK/SL'\,.\,SK'$. Now $1/SL+1/SK=2/l$ (p. 338), where l is the semi-latus rectum; hence $SL\,.\,SK/SL'\,.\,SK'=LK/L'K'$.

Ex. 3. To prove the theorem, for a parabola, that

$$QV^2=4SP\,.\,PV,$$

by using the Rectangle Theorem.

Let the tangent at P (Fig. 142) meet the axis in T and the parallel

through Q to the axis in R; let N, G be the feet of the ordinate and normal at P.

Then $$\frac{RP^2}{RQ \, . \, R\infty} = \frac{TP^2}{TA \, . \, T\infty} \quad \text{or} \quad \frac{QV^2}{PV} = \frac{TP^2}{TA};$$

because $R\infty / T\infty = 1$, since RQ and TA are parallel.

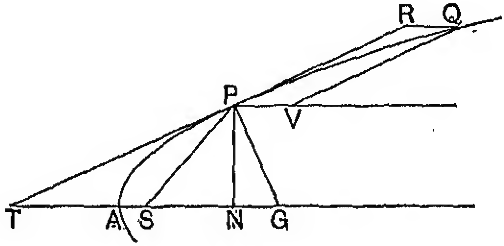


FIG. 142.

But $TA = AN$ and $ST = SP = SG$; therefore

$$TP^2 = TN \, . \, TG = 2TA \, . \, 2SP \quad \text{or} \quad \frac{TP^2}{TA} = 4SP.$$

Hence $$\frac{QV^2}{PV} = 4SP \quad \text{or} \quad QV^2 = 4SP \, . \, PV.$$

Ex. 4. Discuss the application of the Rectangle Theorem to the asymptotes of a hyperbola.

Let CPV in Fig. 127 be a diameter of a hyperbola which bisects the chord QQ' at V, let the tangent at P and QQ' meet the asymptotes in T, T'' and R, R', and let CD be the semi-diameter conjugate to CP.

Then CTR meets the curve at infinity, so that

$$\frac{TP^2}{T\infty^2} = \frac{CD^2}{C\infty^2} \quad \text{or} \quad PT = CD \quad \text{since } T\infty / C\infty = 1.$$

Similarly $PT'' = CD$, and therefore the tangent TT'' is bisected at P.

Also $$\frac{RQ \, . \, RQ'}{R\infty^2} = \frac{TP^2}{T\infty^2} \quad \text{or} \quad RQ \, . \, RQ' = TP^2 = CD^2.$$

Since TT'' is bisected at P, RR' is bisected at V, so that also $RQ \, . \, QR' = PT^2 = CD^2$.

Ex. 5. If a conic cut the sides BC, CA, AB of a triangle ABC in P_1 and P_2, Q_1 and Q_2, R_1 and R_2 respectively, then

$$(P_1)(P_2)(Q_1)(Q_2)(R_1)(R_2) = 1,$$

where (P_1), etc., stand for BP_1/P_1C, etc.; and if a conic touch the sides of the triangle at P, Q, R respectively, AP, BQ, CR are concurrent.

If A, B, C have coordinates (x_1y_1), (x_2y_2), (x_3y_3) and the equation of the conic be $f(x, y)=0$, then

$$\frac{BP_1 \cdot BP_2}{CP_1 \cdot CP_2}=\frac{f(x_2, y_2)}{f(x_3, y_3)}; \quad \frac{CQ_1 \cdot CQ_2}{AQ_1 \cdot AQ_2}=\frac{f(x_3, y_3)}{f(x_1, y_1)}; \quad \frac{AR_1 \cdot AR_2}{BR_1 \cdot BR_2}=\frac{f(x_1, y_1)}{f(x_2, y_2)};$$

hence, multiplying up, we get $(P_1)(P_2)(Q_1)(Q_2)(R_1)(R_2)=1$.

If P_1 and P_2, Q_1 and Q_2, R_1 and R_2 coincide at P, Q, R, the conic touches the triangle at P, Q, R, and

$$\frac{BP^2}{PC^2} \cdot \frac{CQ^2}{QA^2} \cdot \frac{AR^2}{RB^2}=1 \quad \text{or} \quad \frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB}=\pm 1.$$

If the negative sign holds, P, Q, R are collinear by Menelaus's Theorem, and a line would cut the conic in three points, which is impossible. Hence $\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB}=+1$, and AP, BQ, CR are concurrent by Ceva's Theorem.

164. The Intersections of a Conic and a Circle. Circle of Curvature. The intersections of a conic and a circle are of special interest, and their properties are easily investigated by means of the Rectangle Theorem. Let a circle cut a conic in P, Q, P', Q' (Fig. 143), and let PQ, $P'Q'$ meet at O.

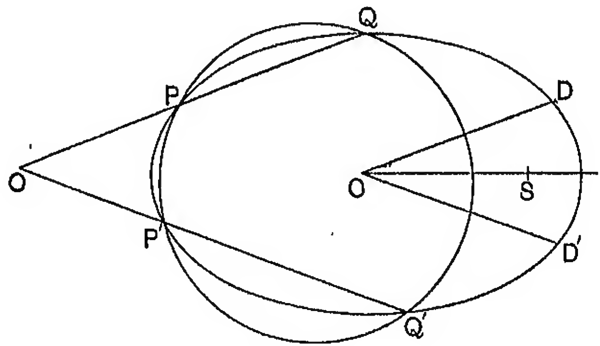


FIG. 143.

Let CD, CD' be semi-diameters of the conic parallel to OPQ, $OP'Q'$. Then

$$\frac{OP \cdot OQ}{OP' \cdot OQ'}=\frac{CD^2}{CD'^2}.$$

But $OP \cdot OQ=OP' \cdot OQ'$; therefore $CD=CD'$. Hence CD,

CD' are equally inclined to the axes of the conic, so that OPQ, $OP'Q'$ are also equally inclined to the axes of the conic. (In the case of the parabola, or indeed any of the conics, we may substitute the ratio of parallel focal chords for CD^2/CD'^2, with the same result.)

Hence the important theorem: *The lines joining the points of intersection of a conic and a circle, taken in pairs, are equally inclined to the axes of the conic.* If the centre of the circle lies on the normal at P to the conic and the circle passes through P, the circle will touch the conic at P, the line joining the pair of intersections at P becomes the tangent at P, so that the tangent at P and the line QR joining the other points of intersection of the circle and conic are equally inclined to the axes. If the centre of this circle move on the normal at P till Q also coincides with P, then *three* of the intersections of the circle and conic coincide at P, and the circle and conic cut elsewhere at R. In this case the tangent at P and the common chord PR of the circle and conic are equally inclined to the axes of the conic (Fig. 144). This circle is the **circle of curvature**; it lies closest of all circles to the conic at P; the centre and radius of the circle are the **centre and radius of curvature.** Hence the following important theorem: *To construct the circle of curvature at a point P on a conic, draw the tangent PT at P* (Fig. 144), *and through P draw a second line cutting the conic again in R so that PT and PR form an isosceles triangle with the axis of the conic; the circle touching PT at P and passing through R is the circle of curvature.* PR is called the (common) **chord of curvature** at P on the conic.

165. Worked Examples on Curvature. In the following examples the properties of the circle of curvature of a conic are investigated.

Ex. 1. **The length of the common chord of curvature at a point θ on an ellipse is**

$$2CD \sin 2\theta,$$

and the radius of curvature is $\dfrac{CD^3}{ab}$.

Let PR (Fig. 144) be the chord of curvature and PT the tangent at P.

Then, according to Theorem 5, § 154, we may write the equation of PR in the form

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{b\cos\theta}=\frac{r}{CD}=k, \text{ say, } \qquad \text{.................(1)}$$

so that $\quad x=a(\cos\theta+k\sin\theta) \quad$ and $\quad y=b(\sin\theta+k\cos\theta)$.

If x, y are the coordinates of R, then $x^2/a^2+y^2/b^2=1$; therefore

$$(\cos\theta+k\sin\theta)^2+(\sin\theta+k\cos\theta)^2=1,$$

so that $\quad k^2+4k\sin\theta\cos\theta=0, \quad$ or $\quad k=-2\sin 2\theta,$

by dropping the zero value of k.

Therefore $\quad \dfrac{r}{CD}=-2\sin 2\theta \quad$ or $\quad PR=2CD\sin 2\theta.$

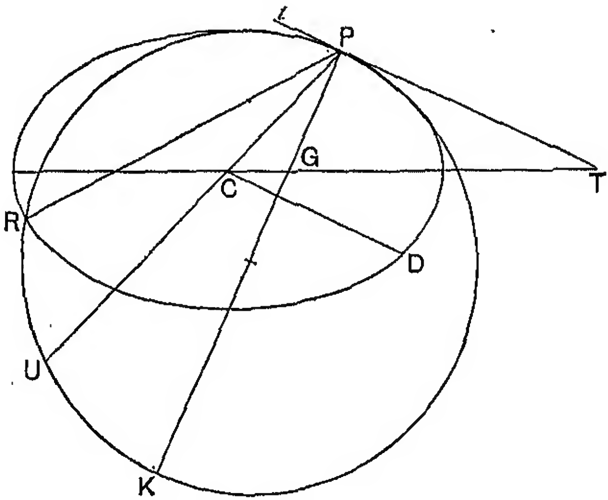


FIG. 144.

Let PK be the diameter of curvature; then

$$\frac{PR}{PK}=\sin PKR=\sin tPR=\sin 2PTC=2\sin PTC\,.\cos PTC.$$

Now, from equation (1), $\sin PTC$ and $\cos PTC$ are numerically equal to $b\cos\theta/CD$ and $a\sin\theta/CD$ respectively, and therefore

$$\therefore \quad \frac{PR}{PK}=2\frac{b\cos\theta}{CD}\cdot\frac{a\sin\theta}{CD}=\frac{ab\sin 2\theta}{CD^2}.$$

But $PR=2CD\sin 2\theta$; therefore

$$PK=\frac{2CD^3}{ab},$$

or the radius of curvature is equal to CD^3/ab.

Ex. 2. **The chord of curvature through the centre of the ellipse is**

$$\frac{2CD^2}{CP}.$$

Let PCU (Fig. 144) be the chord of curvature through C and PK the diameter of curvature.

Then $\qquad PU=PK\cos KPU=PK\sin PCD.$

But $\qquad PK=2CD^3/ab \quad$ and $\quad CP.CD\sin PCD=ab$;

therefore $\qquad PU=\dfrac{2CD^2}{CP}.$

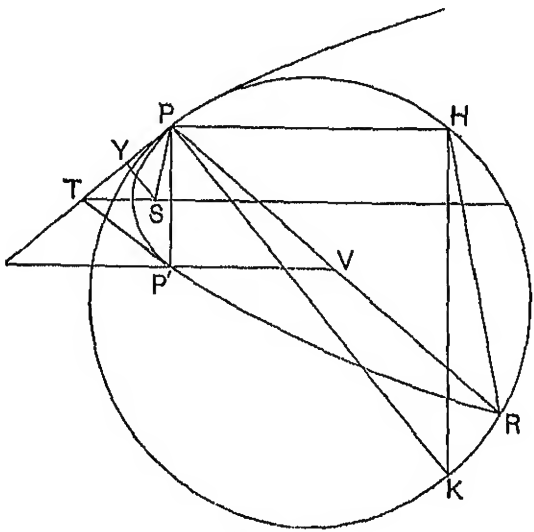


FIG. 145.

Ex. 3. **The length of the common chord of curvature at a point P on a parabola is 4PT, the length of the chord of curvature parallel to the axis is 4SP and the radius of curvature is 2SP²/SY.**

Let PP' (Fig. 145) be the double ordinate through P; then TP' is the tangent at P'. Let PR be the common chord of curvature; then

TP' is parallel to the chord PR. Hence the diameter th bisects PR, so that $PR=4PT$.

If PH is the chord of curvature parallel to the axis, the HPR, SPT are similar; but $PR=4PT$, therefore $PH=4SP$.

If PK is the diameter of curvature, the triangles KPH a are similar; therefore

$$\frac{PK}{SP}=\frac{PH}{SY} \quad \text{or} \quad PK=\frac{4SP^2}{SY}.$$

166. Contact of Conics. Systems of Conics. The curvature is an example of one conic having thr contact with another. We proceed to discuss tl other kinds of contact of two conics.

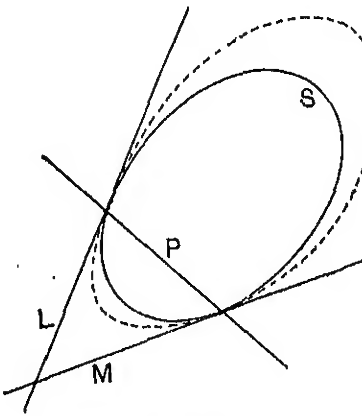


FIG. 146.

Let S stand for $ax^2+2hxy+by^2+2gx+2fy+c$, $S=0$ represents a conic, which will be referred to as t S. Let P, L, M be linear functions of x, y, so tha $L=0$, $M=0$ represent straight lines, which will be to as the lines P, L, M. The intersections of L and M will be denoted by (P, L), (P, M).

Let L and M be *tangents* to the conic S, let P *chord of contact*, and let us consider the contact of S

$$S=\lambda \, . \, LM, \quad \ldots\ldots\ldots\ldots\ldots\ldots$$

where λ is a variable parameter.

The equation (1) represents a conic, since it is

second degree in x, y. The solutions of $S=0$ and $S=\lambda . LM$ (1) regarded as simultaneous equations are the same as those of $S=0$ and $L=0$ together with those of $S=0$ and $M=0$. The solutions of $S=0$ and $L=0$ give the point (P, L) *twice*, since L touches S; and the solutions of $S=0$ and $M=0$ give the point (P, M) *twice*, since M touches S. Hence S meets (1) at (P, L) twice and (P, M) twice; so that S and (1) have L and M for common tangents; S and (1) are said to have *double contact*. The dotted curve in Fig. 146 represents (1).

Again consider the contact of $S=0$ with the conic

$$S=\lambda . PL. \qquad\qquad (2)$$

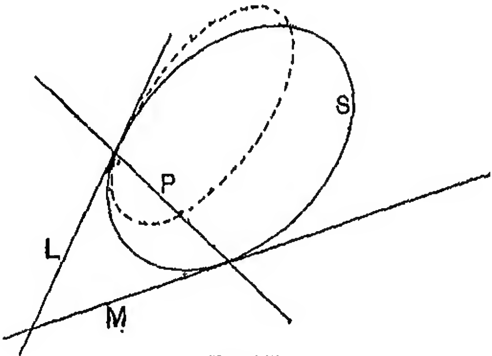


FIG. 147.

The simultaneous solutions of $S=0$ and $S=\lambda . PL$ are those of $S=0$ and $P=0$ with those of $S=0$ and $L=0$; the first of these gives (P, L) and (P, M), the second gives (P, L) twice. Hence S meets the conic (2) at (P, L) *thrice* and at (P, M) *once*. S and (2) have *three-point contact* at (P, L); they are said to *osculate* at (P, L); the circle of curvature is an example. The dotted curve in Fig. 147 represents a conic osculating S at (P, L).

Ex. 1. **Find the equation of the circle of curvature at the point θ on the ellipse $x^2/a^2+y^2/b^2=1$.**

Let L be the tangent to the conic at θ and let P be the common chord of the circle and conic. We know that L and P are equally inclined

to the major axis and that the equations of L and P may therefore be written in the forms

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1 \quad\text{and}\quad y-b\sin\theta=\frac{b\cos\theta}{a\sin\theta}(x-a\cos\theta),$$

or $bx\cos\theta+ay\sin\theta-ab=0$ and $bx\cos\theta-ay\sin\theta-ab\cos 2\theta=0$.

Substituting accordingly in (2), we get

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=\lambda(bx\cos\theta-ay\sin\theta-ab\cos 2\theta)(bx\cos\theta+ay\sin\theta-ab) \quad \text{(i)}$$

as the equation of a conic having 3 point contact with the ellipse at the point θ.

It remains to determine λ so that equation (i) may represent a circle. Now the terms in xy vanish from the equation whatever be the value of λ; the coefficients of x^2 and y^2 are equal if

$$\frac{1}{a^2}-\lambda b^2\cos^2\theta=\frac{1}{b^2}+\lambda a^2\sin^2\theta \quad\text{or}\quad \lambda=\frac{b^2-a^2}{a^2b^2}\cdot\frac{1}{a^2\sin^2\theta+b^2\cos^2\theta} \ldots \text{(ii)}$$

Hence the equation of the circle of curvature is got by substituting in (i) the value of λ given in (ii); we get finally

$$x^2+y^2-2\frac{a^2-b^2}{a}x\cos^3\theta+2\frac{a^2-b^2}{b}y\sin^3\theta=(2b^2-a^2)\cos^2\theta+(2a^2-b^2)\sin^2\theta.$$

Next consider the contact of $S=0$ with the conic

$$S=\lambda\,.\,P^2 \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The simultaneous solutions of $S=0$ and $S=\lambda\,.\,P^2$ are those of $S=0$ and $P=0$ twice over, that is (P, L) and (P, M) twice over. Hence (3), like (1), represents a family of conics having double contact with S at (P, L) and (P, M). The dotted curve of Fig. 140 would also represent (3).

Ex. 2. A point moves so that the tangent from the point to a fixed circle bears a constant ratio to the perpendicular from the point to a fixed line; show that the locus of the point is a conic which touches the circle at the points of intersection of the line and the circle, and that the eccentricity of the conic is the value of the constant ratio.

Let $x^2+y^2=r^2$ and $x=k$ be the equations of the line and circle, and let λ be the constant ratio; then the defining condition gives as the equation of the locus

$$x^2+y^2-r^2=\lambda^2(x-k)^2, \ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

which, according to equation (3), represents a conic having double contact with the circle $x^2+y^2-r^2=0$ at the points of intersection of the circle and the line $x-k=0$.

To show that λ is the eccentricity of the conic, we write (iii) in the canonical form: we get

$$\left(x+\frac{k\lambda^2}{1-\lambda^2}\right)^2+\frac{y^2}{1-\lambda^2}=\frac{\lambda^2k^2+r^2}{1-\lambda^2}+\frac{k^2\lambda^4}{(1-\lambda^2)^2}.$$

Hence, if e is the eccentricity,

$$e^2=1-(1-\lambda^2) \quad \text{or} \quad e=\lambda.$$

Ex. 3. **To find the equation of the pair of tangents to a given conic from a given point.**

Let $ax^2+2hxy+by^2+2gx+2fy+c=0$ be the given conic and (x_1, y_1) the given point.

By equation (3),

$$ax^2+2hxy+by^2+2gx+2fy+c=\lambda\{axx_1+h(xy_1+x_1y)+byy_1+g(x+x_1) \\ +f(y+y_1)+c\}^2 \quad \text{(iv)}$$

represents a conic having double contact with the given conic at the intersections of the given conic and the polar of (x_1, y_1).

But the pair of tangents from (x_1, y_1) is such a conic, which also then passes through the point (x_1, y_1); λ is thus determined by putting x_1, y_1 for x, y in equation (iv). We get finally

$$(ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c)(ax^2+2hxy+by^2+2gx+2fy+c) \\ =\{axx_1+h(xy_1+x_1y)+byy_1+g(x+x_1)+f(y+y_1)+c\}^2.$$

We next consider the contact of the conic S with the conic

$$S=\lambda \,.\, L^2. \quad \text{..........................(4)}$$

The simultaneous solutions of $S=0$ and $S=\lambda \,.\, L^2$ are those of $S=0$ and $L=0$ twice over, that is (P, L) *four times*. S and (4) are said to have 4-point contact at (P, L); the dotted curve in Fig. 148 represents (4).

2-point, 3-point, 4-point contact are often spoken of as contact of the first, second, third order.

Ex. 4. The locus of the centre of a variable conic having 4-point contact with a fixed conic at a fixed point on it is a straight line.

Refer the fixed conic to the tangent and normal at the fixed point on it as axes of x and y, and so write the equation of the fixed conic in the form

$$y=ax^2+2hxy+by^2.$$

Then, by equation (4),

$$y-ax^2-2hxy-by^2=\lambda y^2 \quad \text{..........................(v)}$$

is the equation of the variable conic.

For the centre of (v) we have

$$ax+hy=0 \quad \text{and} \quad 2hx+2(b+\lambda)y=1,$$

so that the locus of the centre is the line $ax+hy=0$.

Now take *any* three lines L, M, P and consider th equation

$$P^2 = \lambda . LM. \quad\quad (5$$

Clearly (5) is a conic; $L=0$ meets it at (P, L) twice an $M=0$ meets it at (P, M) twice, so that (5) represents variable conic touching L and M at (P, L) and (P, M)

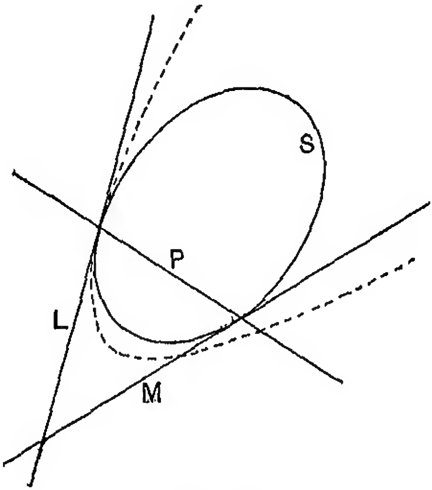


FIG. 148.

Let $S=0$ be this conic; then the equations $S=0$ an $P^2=\lambda . LM$ represent one and the same conic. Hence

A conic may be defined as the locus of a point whic moves so that the square of its distance from one of thre fixed straight lines is proportional to the product of it distances from the other two; these two are tangents to th conic, and the remaining line is their chord of contact.

Ex. 5. Find the equation of the parabola which touches the axes o x and y where the line $ax+by=1$ meets them.

Equation (5), $P^2=\lambda LM$, gives any conic which meets the line L, M twice at each of the points (P, L), (P, M). Hence the equatio of the parabola (Fig. 149) may be written

$$(ax+by-1)^2=\lambda . xy. \quad\quad (vi$$

This equation represents a parabola if the terms of the second degree form a perfect square; therefore

$$a^2x^2+(2ab-\lambda)xy+b^2y^2$$

is a perfect square, so that

$$(2ab-\lambda)^2=4a^2b^2 \quad \text{or} \quad \lambda=4ab,$$

since the value $\lambda=0$ gives the pair of straight lines $(ax+by-1)^2=0$.

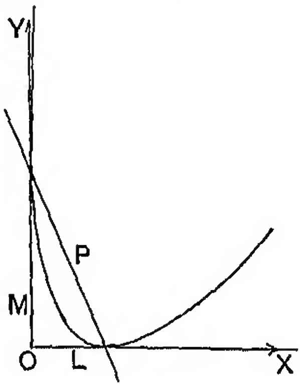


FIG. 149.

The equation of the parabola is therefore, by (vi),

$$(ax+by-1)^2=4abxy \quad \text{or} \quad ax\pm 2\sqrt{abxy}+by=1,$$

which may be written $\quad \sqrt{(ax)}+\sqrt{(by)}=1.$

Finally, take any four lines L, M, P, Q and consider the equation

$$PQ=\lambda\,.\,LM. \qquad \text{.........................(6)}$$

It will now be clear that (6) represents a conic which passes through the four points (P, L), (P, M), (Q, L), (Q, M).

Ex. 6. Show that in general a conic can be drawn through five points, and a one-fold infinity of conics through four points, and that the locus of the centres of these is a conic.

Let A, B, C, D be four points. Choose axes so that A, B lie on the x-axis and C, D on the y-axis, and let the coordinates of A, B, C, D be $(a, 0)$, $(b, 0)$, $(0, c)$, $(0, d)$ respectively.

Then, by (6), the equation

$$\left(\frac{x}{a}+\frac{y}{c}-1\right)\left(\frac{x}{b}+\frac{y}{d}-1\right)=\lambda\,.\,xy \qquad \text{....................(vii)}$$

represents a conic through A, B, C, D for all values of λ.

The coordinates of a fifth point E will serve to determine λ so that one conic can be drawn through A, B, C, D, E, while a one-fold infinity of conics can be drawn through A, B, C, D, one conic for each value of λ.

For the centre of (vii) we have

$$\frac{2}{ab}\cdot x+\left(\frac{1}{ad}+\frac{1}{bc}-\lambda\right)y-\left(\frac{1}{a}+\frac{1}{b}\right)=0,$$

$$\left(\frac{1}{ad}+\frac{1}{bc}-\lambda\right)x+\frac{2}{cd}\cdot y-\left(\frac{1}{c}+\frac{1}{d}\right)=0,$$

so that the locus of the centre is given by the equation

$$\frac{2}{ab}\cdot x^2-\frac{2}{cd}\cdot y^2=\left(\frac{1}{a}+\frac{1}{b}\right)x-\left(\frac{1}{c}+\frac{1}{d}\right)y,$$

which represents a conic passing through the intersections of AB and CD, AC and BD, AD and BC.

EXERCISES LII.

1. T is a variable point on the tangent at P on a parabola, and the diameter through T meets the curve in Q; show that TP^2 is proportional to TQ.

2. PQ is a chord of a parabola, and the diameter through R, a point in PQ, meets the curve in O and the tangent at P in T; show that

$$TO : OR = PR : RQ.$$

3. C is the centre of an ellipse, PCP' and DCD' a pair of conjugate diameters. PEP' is a semi-circle on PCP' as diameter, MQ is an ordinate of PCP', and MR and CE, each perpendicular to PCP', meet the semi-circle in R and E respectively; show that QR is parallel to DE.

4. Two conics, whose centres are C and C', cut in four points. CA, CB and $C'A'$, $C'B'$ are the semi-diameters parallel to a pair of common chords. Prove that AA', BB', CC' are concurrent.

5. Parallel tangents at Q, Q' on a conic meet a third tangent drawn at P on the conic in T, T' respectively; show that

$$\frac{PT}{PT'}=\frac{QT}{Q'T'}.$$

6. Two conics $S_1=0$ and $S_2=0$ intersect in four points A, B, C, D; show that the equation of any conic through A, B, C, D can be written in the form $S_1=kS_2$. If R is a variable point on a third conic through A, B, C, D, and a line through R parallel to a fixed direction meet S_1 in P_1, P_2, and S_2 in Q_1, Q_2, prove that $RP_1 . RP_2 : RQ_1 . RQ_2$ is a constant ratio.

7. If TP, TQ are tangents to an ellipse, and the normals PG, QK meet the axis at G, K, prove that $TP/PG=TQ/QK$.

8. The foot N of the ordinate of a point P on a parabola is the centre of the circle of curvature at its vertex. Prove that the centre of the circle of curvature at P lies on the parabola.

9. Prove that the focal chord of curvature at any point of a conic is equal to the focal chord of the conic parallel to the tangent at that point.

10. Prove that the common chord of a parabola and its circle of curvature at any point constantly touches another parabola having the same vertex.

11. If the radius of curvature at P on an ellipse is twice PG, where G is the point where the normal at P meets the axis, prove that $CP = CS$.

12. Find a point on a parabola at which the focal chord of curvature is also a chord of the parabola.

13. Show that the locus of the second point of intersection of the normal at P to the parabola $y^2 = 4ax$ with the circle of curvature is the curve
$$125ay^2 = 4(x-4a)(2x-3a)^2.$$

14. If O is the centre of curvature for the vertex A of an ellipse and the normal at P meets the major axis in G, then GA, CA and the perpendiculars from O and C on the tangent at P are proportionals.

15. O is the middle point of PQ, the normal chord at P to a rectangular hyperbola; show that O is the centre of curvature at P.

16. Prove that the chord of curvature through a focus of a conic is $2CD^2/CA$, where CD, CP are conjugate semi-diameters and P is the point of contact of the circle of curvature.

17. If PR is the chord of curvature at the point P on an ellipse whose eccentric angle is θ, prove that -3θ is the eccentric angle of R.

18. If r and r' are the chords of curvature at the extremities of CP, CD, a pair of conjugate semi-diameters of the ellipse
$$x^2/a^2 + y^2/b^2 = 1,$$
prove that
$$r^2 + r'^2 = 4(a^2+b^2)\sin^2 2\theta,$$
where θ is the eccentric angle of one of the extremities of the semi-diameters.

19. If two tangents OT, OT' are drawn from the point (f, g) to the ellipse $x^2/a^2 + y^2/b^2 = 1$ and d, d' are the parallel semi-diameters, show that
$$\frac{OT^2}{d^2} = \frac{OT'^2}{d'^2} = \frac{f^2}{a^2} + \frac{g^2}{b^2} - 1.$$

20. If A and B be points on the axes of x and y such that $OA = a$, $OB = b$, prove that the equation
$$xy + (Bx + Cy)(x/a + y/b - 1) = 0$$
represents a conic circumscribed about triangle OAB, and that $Bx + Cy = 0$ is a tangent to the conic. Determine B and C so that the conic may become the circle OAB.

21. Prove that the equation of a parabola which touches the two straight lines $ax^2+2hxy+by^2=0$, where they are cut by the line $lx+my+1=0$, is

$$(ax^2+2hxy+by^2)(am^2-2hlm+bl^2)=(ab-h^2)(lx+my+1)^2.$$

22. The equation of the family of conics inscribed in the rectangle formed by the lines $x\pm a=0$, $y\pm b=0$ is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+2\lambda\,.\,\frac{xy}{ab}+\lambda^2=0.$$

Prove also that the locus of the foci is $x^2-y^2=a^2-b^2$.

23. Show that a conic may be defined as the locus of a point P such that OP^2 is proportional to $PM\,.\,PN$, when O is a fixed point and PM, PN the perpendiculars from P on two fixed straight lines.

24. If $u=0$ is the equation of a conic, and $v=0$ the equation of its director circle, show that the equation $u-\lambda v=0$ represents for one value of λ, the directrices of the conic.

25. Tangents TP, TQ are drawn from $T(x_1y_1)$ to the parabola $y^2=4ax$; prove that the equation of the circle circumscribing the triangle TPQ is $a(x^2+y^2)-(y_1^2+2a^2)x-y_1(a-x_1)y+ax_1(2a-x_1)=0$.

26. If $S=0$ and $S'=0$ be the equations of two conics, and $L=0$ and $M=0$ the equations of two straight lines, interpret the equations $S-kS'=0$; $S+kLM=0$, $S+kL^2=0$, when k is a constant.

27. Prove that the product of the perpendiculars let fall from any point of a conic on two opposite sides of an inscribed quadrilateral is in a constant ratio to the product of the perpendiculars let fall on the other two sides.

28. Prove that the equation of a circle which touches the parabola $y^2=4ax$ and passes through its focus may be written

$$(1+m^2)(y^2-4ax)+(x-my+am^2)(x+my+3a)=0.$$

29. Prove that the equation $\sqrt{ax}+\sqrt{by}=1$ represents a parabola, and show that the length of its latus rectum is

$$\frac{4ab}{(a^2+b^2)^{\frac{3}{2}}}.$$

If $a+b=k$, where k is a constant, show that the locus of the foci of all such parabolas is a circle.

30. A hyperbola touches the axis of y at the origin, and the line $y=7x-5$ at the point $(1, 2)$. One of the asymptotes is parallel to the axis of x. Find the equation of the curve.

31. Find the equation of the conic through the points of intersection of $3x^2+y^2=4$ and $5xy-y^2=2$, and through the point $(-2, 3)$.

32. Show that the equation

$$\frac{A}{x\cos\alpha_1+y\sin\alpha_1-p_1}+\frac{B}{x\cos\alpha_2+y\sin\alpha_2-p_2}+\frac{C}{x\cos\alpha_3+y\sin\alpha_3-p_3}=0$$

represents a conic circumscribing the triangle formed by the lines $x\cos\alpha_1+y\sin\alpha_1-p_1=0$, etc., where A, B, C are any constants, and that the conic is the circumscribing circle if

$$A:B:C=\sin(\alpha_2-\alpha_3):\sin(\alpha_3-\alpha_1):\sin(\alpha_1-\alpha_2).$$

33. Show that $S=kS'$, where k is a variable parameter, represents the system of coaxal circles to which belong the circles $S=0$, $S'=0$. Prove that tangents from any point on a fixed circle of a coaxal system to two other fixed circles of the system are in a given ratio.

34. Prove that two parabolas can be drawn through four given points.

35. If a variable conic pass through three fixed points and have an asymptote parallel to a given line, the locus of its centre is a parabola. If it pass through two given points and have its asymptotes parallel to two given lines, the locus of its centre is a straight line.

36. Find the locus of the centres of all rectangular hyperbolas having contact of the third order with the ellipse $x^2/a^2+y^2/b^2=1$.

37. PQ is the common chord of a parabola $y^2=4ax$ and its osculating circle. Show that the locus of the intersection of PQ with the perpendicular drawn to it from the vertex is $y^2(3a-x)=x^3$.

38. The polar of the focus of the parabola $y^2=4ax$ with respect to any rectangular hyperbola which has 4-point contact with the parabola touches the parabola $$y^2=4a(3x+2a).$$

39. A rectangular hyperbola passes through the three points $(b, 0)$, $(c, 0)$, $(0, a)$; show that it meets the y-axis again at the point whose ordinate is $-bc/a$, and deduce that if a rectangular hyperbola pass through the vertices of a triangle, it passes through the orthocentre.

40. Using the relation between the eccentricity and the angle between the asymptotes of a conic, find an equation giving the eccentricity of a conic specified by the general equation.

41. If through a given point on a conic two lines be drawn which make with the normal angles the product of whose tangents is constant, show that the join of their extremities passes through a fixed point on the normal.

42. If through any point on an equilateral hyperbola be drawn two chords at right angles, the perpendicular let fall on the line joining their extremities is the tangent to the curve.

43. If a circle have double contact with a conic, the tangent drawn to the circle from any point on the conic is in a constant ratio to the perpendicular from the point on the chord of contact.

44. If two conics have double contact, the square of the perpendicular from any point of one upon the chord of contact is in a constant ratio to the rectangle under the segments of that perpendicular made by the other.

45. If two conics have each double contact with a third, their chords of contact with the third conic, and a pair of their chords of intersection with each other, will all pass through the same point and form a harmonic pencil.

46. The chords of contact of two conics with their common tangents pass through the intersection of a pair of their common chords.

47. If three conics have each double contact with a fourth, six of their chords of intersection will pass three by three through the same points.

48. If three conics have one chord common to all, their three other common chords will pass through the same point.

49. If four points on a conic are given, its chord of intersection with a fixed conic passing through two of these points will pass through a fixed point.

50. Two conics S_1 and S_2 intersect in the four points A, B, C, D. Lines AF_1F_2, BG_1G_2 are drawn cutting S_1 and S_2 in F_1, G_1 and F_2, G_2 respectively; prove that the intersection of F_1G_1 and F_2G_2 lies on the line CD.

ANSWERS.

§ 3, p. 5. **1.** $2, 6, -5, -3$. **3.** $5, -1, 1, -5, (a+b)/2$.

§ 4, p. 6. **1.** (i) $\frac{3}{2}$; (ii) $-\frac{8}{3}$; (iii) $\frac{7}{2}$; (iv) $-\frac{1}{4}$; (v) $\frac{3}{2}$.

Exercises I. p. 7.

1. 4. **2.** $-1, 1, -3, -1, -4$. **3.** (i) $\frac{11}{3}, \frac{16}{3}$; (ii) $-1, 2$; (iii) $-2, -3$.
4. $\frac{13}{5}, -7, -9\frac{3}{5}$. **5.** -6. **6.** $-2, 6$.
7. $-\frac{1}{2}, 0$. **8.** 11. **9.** $\frac{2}{3}(a+b)$.
13. For first part take M as origin, for second part take A as origin.
14. P, $\frac{1}{2}[(x_1+x_2-d) \pm \sqrt{\{(x_1-x_2)^2+d^2\}}]$;
Q, $\frac{1}{2}[(x_1+x_2+d) \pm \sqrt{\{(x_1-x_2)^2+d^2\}}]$.
15. $2mn(x_2-x_1)/(m^2-n^2)$. **16.** $(m-n)(x_1-x_2)/(1+m)(1+n)$.

Exercises II. p. 10.

1. (1) $+4$ in. per sec.; (2) $x=3$; (3) $x=15$, $x=-9$; (4) $\frac{1}{2}$ sec. before zero-time, 1 sec. before zero-time.
2. $x=1+2t$. **3.** $x=2-3t$. **6.** $-1, 4, 5$.
7. $\frac{7}{5}$. **8.** $3\frac{3}{7}$. **9.** $x_1+(x_2-x_1)t/(2t-1)$.

§ 9, p. 16. **1.** (1) 5; (2) 5; (3) 13.
2. (1) $-2, 2$; (2) $-3, \frac{3}{2}$; (3) $1, \frac{1}{2}$; (4) $-3, \frac{3}{2}$. **4.** $\frac{25}{2}$.
5. 10 or -14. **7.** $\sqrt{17}, \sqrt{10}, 4, \sqrt{5}$; $\sqrt{13}, \sqrt{34}$.

Exercises III. p. 18.

2. $(-1, 2)$. **3.** $(\frac{8}{3}, \frac{11}{3})$, $(\frac{10}{3}, \frac{13}{3})$.
4. $(2, 1)$; $(3, 0)$; $(11, -8)$; $(23, -20)$. **5.** $(\frac{17}{5}, \frac{13}{5})$, $(25, -7)$, $\frac{12}{5}\sqrt{97}$.
6. $2\sqrt{130}$, $3\sqrt{130}$. **7.** $\frac{20}{13}, -\frac{10}{11}, \frac{12}{11}\sqrt{130}$.
8. $\frac{425}{6}, \frac{5}{3}\sqrt{85}, \frac{15}{2}\sqrt{85}$. **9.** $(\frac{5}{3}, -\frac{10}{3})$.

§ 14, p. 21. **1.** $(\frac{1}{3}, \frac{5}{3})$, $(-\frac{25}{2}, 0)$.

Exercises IV. p. 26.

4. $y+3=0$. **6.** No. **17.** $(-2, -2\cdot5)$, $(-5, -4)$.

19. $2x-y=1$. **21.** $(1, -1)$, $(-2, -3)$, $(7, 3)$ lie on the line.

Exercises V. p. 32.

6. $\frac{3}{4}$, $-\frac{7}{8}$, $\frac{5}{3}$, $\frac{1}{3}$, $-\frac{1}{5}$. **15.** $(13, 4)$.

22. The line through $(3, 1)$ of gradient $1/2$.

24. (1) line through $(0, 3)$ of gradient $\frac{1}{2}$; (2) line through $(0, 2)$ of gradient $\frac{1}{3}$; (3) line through $(0, 1)$ of gradient $\frac{1}{2}$; (4) line through $(0, 2)$ of gradient $-\frac{3}{4}$; (5) line through $(0, -3)$ of gradient -2.

26. $(2, -1)$, $(3, 2)$ are on the line. **28.** $3x=2y$. **29.** $y=mx+c$.

30. $2x+3y=19$, $x-2y=1$, $3x-2y+16=0$, $2x+y=7$.

32. The gradient of BC is $\frac{1}{2}$, therefore the gradient of the perpendicular from A is -2. Hence required equation is $\frac{y-5}{x-2}=-2$ or $2x+y=9$.

33. $-\frac{9}{2}$, $-\frac{1}{2}$, $\frac{5}{6}$; $2x-9y+28=0$, $2x-y+5=0$, $6x+5y=8$.

34. $7x+6y=61$, $11x-6y+43=0$, $x-3y=28$. **35.** 4, $4\sqrt{13}$.

36. $5\sqrt{2}$. **40.** $(1, 5)$ or $(5, -1)$. **41.** $(a-b, a+b)$ or $(a+b, b-a)$.

42. $(a+b-d, b+c-a)$ or $(a+d-b, a+b-c)$.

Exercises VI. p. 42.

1. $44\cdot5$. **2.** $-7\cdot5$. **3.** $-38\cdot5$. **4.** $\frac{1}{2}(bx+ay-ab)$.

5. 72. **6.** -39. **7.** 28. **8.** -3.

9. 22. **12.** $4x-9y+37=0$.

Exercises VII. p. 45.

3. $3x+4y=6$. **4.** $4x-6y=15$.

5. $\sqrt{3}x-3y+6=0$. **6.** $5\sqrt{3}x+5y+11=0$.

8. $2, -2, -3/4, 1/2, -2/3, 1/2, -3/7, 3/4, -7/5$.

9. $-a/b$. **10.** $-b/a$. **11.** m, m.

Exercises VIII. p. 49.

1. $3x+2y+19=0$. **2.** $3x-4y=32$, $3x+4y+8=0$.

3. $2x-5y+11=0$. **4.** $x+y=4$, $x+y+2=0$.

5. $x+y=4$, $x-y+2=0$. **6.** $4x-3y+3=0$.

7. $2, 3, -2, 2/5, -3/2, -a/b$. **10.** $5x-2y=11$, $3x+2y=29$, $x+6y=15$.

11. $x-3y=1$, $17x+12y+4=0$, $22x-3y=1$. **12.** 7, $-\frac{1}{7}$, $x+7y=36$.

13. $4x=9y$. **14.** $3x-5y=8$. No. **15.** (i), (iv). **16.** $(1, 1)$.

17. $\left(-\frac{13}{7}, \frac{100}{49}\right)$. **18.** $\left(\frac{69}{16}, \frac{23}{8}\right)$.

19. $\left(-\frac{97}{86}, -\frac{62}{87}\right)$. **20.** $\left(\frac{17}{8}, \frac{10}{3}\right)$, $(0, -\frac{1}{3})$.

§ 28, p. 51. **2.** (i) $2x-y=7$; (ii) $y=3x-8$; (iii) $5x+4y=3$; (iv) $3x-2y=8$; (v) $y=2x+5$; (vi) $2x=7y$.

Exercises IX. p. 52.

2. (i) $4x+3y=26$; (ii) $5x-2y=13$; (iii) $x+2y=8$; (iv) $3x-y+7=0$; (v) $3x+5y+9=0$; (vi) $2x-7y=12$; (vii) $4x+3y=0$.

3. $7x-2y+30=0$, $2x+7y+1=0$. **4.** $3x+8y+46=0$, $8x-3y+1=0$.

5. $(\frac{29}{11}, \frac{35}{11})$. **6.** $(-\frac{7}{5}, -\frac{9}{5})$; 4.

Exercises X. p. 54.

2. $\frac{14}{5}$. **3.** $\frac{1}{2}\sqrt{2}$. **4.** $\frac{1}{2}\sqrt{2}$. **6.** $\sqrt{2}$, 2, $\frac{15}{26}\sqrt{13}$. **7.** $\frac{11}{5}$. **11.** 10.

Exercises XI. p. 58.

2. $x=3+2t$, $y=5+t$. **5.** $(-\frac{47}{17}, \frac{40}{17})$. **6.** $(\frac{11}{7}, \frac{8}{7})$.

7. $(-1, 2)$. **9.** $x+3y=1$. **10.** $dx-by=ad-bc$.

12. $x=1-3u$, $y=2+2u$; $x=1+2v$, $y=2+3v$. **13.** $(\frac{163}{13}, -\frac{95}{13})$.

§ 32, p. 61. **3.** $5x+3y+15=0$. **4.** $2x-y=4$.

5. $\frac{1}{2}, \frac{1}{3}$; $\frac{4}{3}, -2$; $-\frac{4}{5}, -\frac{4}{7}$; $-\frac{4}{3}, \frac{4}{5}$.

§ 33, p. 64. **3.** $x\frac{5}{13}+y\frac{12}{13}=\frac{8}{13}$; $\tan\alpha=\frac{12}{5}$, $p=\frac{8}{13}$.

4. (i) $\frac{12}{5}$, 1; (ii) $\frac{4}{3}, \frac{54}{25}$; (iii) $-\frac{4}{3}, \frac{7}{5}$; (iv) $-\frac{5}{12}, \frac{10}{13}$; (v) $\frac{3}{2}, \frac{4}{13}\sqrt{13}$; (vi) $\frac{1}{3}, \frac{1}{5}\sqrt{10}$; (vii) $\frac{q}{p}, \frac{r}{\sqrt{p^2+q^2}}$; (viii) $\frac{m}{l}, \frac{n}{\sqrt{l^2+m^2}}$.

§ 34, p. 65. **1.** $3\sqrt{2}$. **2.** $\left(\frac{a+b}{2}, \frac{b-a}{2}\sqrt{3}\right)$, $(b>a)$.

§ 35, p. 68. **8.** 1, 1·5, 5; 45°, 56° 19′, 78° 41′.

9. $x-3y+10=0$, $3x+y=13$.

10. $(y-b)(m+l\tan\alpha)+(x-a)(l-m\tan\alpha)=0$,
$(y-b)(m-l\tan\alpha)+(x-a)(l+m\tan\alpha)=0$.

11. $x-5y+32=0$.

§ 36, p. 69. **2.** $4x+4y=1$, $10x-10y+3=0$. **3.** $x-y=0$, $x+y=0$.

4. $2x-10y=1$, $64x+8y+33=0$.

5. $19x-399y+600=0$, $189x+9y=80$. **6.** $\frac{815}{9}, \frac{5}{87}$.

7. $(3\sqrt{2}-2)x-(\sqrt{2}+4)y+2\sqrt{2}+7=0$,
$(3\sqrt{2}+2)x-(\sqrt{2}-4)y+2\sqrt{2}-7=0$.

§ 37, p. 71. **1.** $(4, -1)$. **2.** $(-2, 1)$. **3.** $(0, 2)$. **4.** $(2, 3)$.

5. $(3, 1)$. **6.** $(3, -2)$. **10.** (b, a). **12.** $(1, 1)$.

§ 36, p. 74. 3. $50x$ [illegible] $61y$.

4. (i) $4x$ [illegible] $y+5$; (ii) $10x+117y$ [illegible]

6. (i) $210x+167$ [illegible] y; (ii) [illegible] 163.

7. (i) $10x$ [illegible] $y+18$, $3x+2y+10$ [illegible] $7x$ [illegible] $7y$;

(ii) $5x+10y$ [illegible], $2x$ [illegible] $3y+0$, $7x+7y$ [illegible] 13.

Exercises XII. p. 75.

1. 2, [illegible]. 2. [illegible]

3. $(ad-bc)/d$, $(bc-ad)/b$.

4. [illegible]

8. b/a, $c/\sqrt{a^2+b^2}$.

10. (i) [illegible]; (ii) $dd'+bb'=0$; (iii) [illegible]

11. [illegible] 12. [illegible]

13. 4. 14. [illegible]; $x+y=2$, [illegible] 16. [illegible]

17. $(-\frac{8}{3}, -\frac{2}{3})$. 22. The lines are concurrent.

20. $64x+49y+10=0$. 24. [illegible]

25. $25x+23y$, $23x+25y=0$; [illegible]

26. $(1, 2)$. 27. [illegible]

29. $a^3+b^3+c^3=3abc$; or, if a, b, and c are real, $a+b+c=0$.

30. $3x+11y=16$. 31. [illegible]

32. $p=2$. 33. $(3, 4)$.

§ 39, p. 80. 3. $(2x+y-4)(3x+5y-2)$ [illegible] 4. [illegible]

5. (i) $x+y=1$, $y=x+1$; (ii) [illegible]; (iii) [illegible]

(iv) $3x+8=2y$, $2x+3y=8$; (v) [illegible]

(vi) $7x+8=9y$, $12x+6y=5$; (vii) [illegible]; (viii) [illegible]

(ix) $a(b-c)x=c(a-b)y$, $x=y$; (x) [illegible]

§ 41, p. 84. 3. Yes. 4. [illegible]

5. [illegible] 7. [illegible]

9. (i) $ax^2+2hxy+by^2=0$; (ii) $bx^2-2hxy+ay^2=0$; or

(i) [illegible]; (ii) [illegible]

§ 42, p. 85. 1. [illegible], $63°8'$. 5. [illegible]

Exercises XIII. p. 94.

5. [illegible] 6. [illegible] 7. [illegible]

9. 12. 12. 38. 13. [illegible]

§ 47, p. 97. 2. [illegible]

Exercises XIV. p. 100.

1. $(-2, -1)$. **2.** $(-3, -1)$. **4.** $ax^2+2hxy+by^2=0$.

8. $\left(\dfrac{hf-bg}{ab-h^2}, \dfrac{gh-af}{ab-h^2}\right)$. **9** $\dfrac{(x-p)^2-(y-q)^2}{a-b}=\dfrac{(x-p)(y-q)}{h}$; the same.

11. $12x-7y=61$. **12.** $4x^2+4xy-3y^2-6x-5y+2=0$.

13. $x-3y+3\sqrt{2}=0$. **14.** $xy+6x+4y=1$.

15. $4x^2y^2=a^2(x^2+y^2)$. **16.** $y(y^2+3x^2)=a^3\sqrt{2}$.

17. $\dfrac{x^2}{b^2}+\dfrac{y^2}{a^2}=1$. **18.** $\xi\eta=a^2\sin\alpha\cos\alpha$.

§ 50, p. 105. **1.** (i) $x^2+y^2=4$; (ii) $x^2+y^2=16$; (iii) $x^2+y^2=25$; (iv) $x^2+y^2+6x-2y+6=0$; (v) $x^2+y^2+4x-4y+4=0$; (vi) $x^2+y^2-4x-2y=4$; (vii) $x^2+y^2=2y$; (viii) $x^2+y^2+2y=3$; (ix) $x^2+y^2-4x=5$; (x) $x^2+y^2+6x=16$; (xi) $x^2+y^2-4x+6y+12=0$; (xii) $x^2+y^2+6x-8y=24$; (xiii) $x^2+y^2-4x+5y+10=0$; (xiv) $18x^2+18y^2+54x+18y+37=0$.

2. $x^2+y^2=25$. **3.** $x^2+y^2+10x+2y=143$.

4. $x^2+y^2-2x+2y=287$. **5.** $(6, 0)$, $(-2, 0)$. **6.** $(0, 13)$, $(0, -11)$.

7. $(1, -1)$, $(-3, -1)$. **8.** $(c+2a, 2b)$, $(-c, 2b)$.

§ 51, p. 108. **2.** (i) $(3, 4)$, 5; (ii) $(-3, -4)$, 4; (iii) $(1, -1)$, 5; (iv) $(-2, 3)$, 4.

3. (i) $(0, 0)$, 2; (ii) $(0, 0)$, 3; (iii) $(1, -2)$, 2; (iv) $(2, 3)$, 4; (v) $(-1, 1)$, 2; (vi) $(\frac{3}{2}, \frac{1}{2})$, 1; (vii) $(-\frac{5}{2}, \frac{3}{2})$, 3; (viii) $(\frac{1}{3}, -\frac{2}{3})$, $\frac{1}{3}\sqrt{5}$; (ix) $(-\frac{1}{2}, -\frac{1}{2})$, $\sqrt{2\cdot 1}$.

Exercises XV. p. 111.

1. $x^2+y^2-6x-2y+5=0$; $\sqrt{5}$, $(3, 1)$. **2.** $x^2+y^2=2x+2y$.

3. $x^2+y^2=2x-6y$. **4.** $(-\frac{1}{4}, \frac{11}{4})$.

5. $x^2+y^2-x+11y+8=0$; $\sqrt{22\cdot 5}$. **6.** $(12, 1)$, $(4, 5)$.

7. $(1, 1)$, $(\frac{7}{5}, \frac{1}{5})$. **9.** $x^2+y^2-10x-y+19=0$.

17. $x^2+y^2=2y$; a circle, centre $(0, 1)$ and radius 1.

18. $x^2+y^2-8x+8=0$; a circle, centre $(4, 0)$ and radius $2\sqrt{2}$.

21. $(2, -1)$, $(-\frac{2}{13}, \frac{29}{13})$. **22.** $(\frac{4}{5}, \frac{13}{5})$, $(2, 3)$. **23.** $5x-3y+5=0$.

§ 53, p. 115. **2.** (i) $2x+3y=18$; (ii) $y=x+2$; (iii) $2x-y=5$; (iv) $2x+3y=21$; (v) $y=x-3$; (vi) $2x-y=7$; (vii) $y=2x+4$; (viii) $4x+6y=15$; (ix) $12x-6y=23$.

§ 56, p. 123. **1.** 4. **2.** 3. **3.** $\frac{1}{2}\sqrt{2}$.

Exercises XVI. p. 124.

1. $x^2+y^2-8x-10y+16=0$; $(0, 8)$.

2. $x^2+y^2-4x-6y+9=0$; $y=5$. **3.** $x^2+y^2-2ax-by+a^2=0$.

4. $x^2+y^2-6x-6y+9=0$, $x^2+y^2-6x+24y+9=0$.

5. $x^2+y^2-2x-2y+1=0$, $x^2+y^2-12x-12y+36=0$,
$x^2+y^2-6x+6y+9=0$, $x^2+y^2+4x-4y+4=0$.

6. $x^2+y^2-4x=77$. **7.** $x^2+y^2-2px=r^2+2ap$.

11. $x\pm\sqrt{35}\,.\,y=30$, $2x\pm\sqrt{5}\,.\,y=15$.

Exercises XVII. p. 128.

2. (i) $x+5=0$; (ii) $5x-5y=1$; (iii) $5x-3y+7=0$; (iv) $6y=-x+10$.

5. $x=0$. **6.** $(0, \pm i\sqrt{6})$. **7.** $(0, \pm 2)$. **8.** $(0, \pm 2i)$.

9. $x=0$. **10.** $x=0$. **11.** $(0, \pm 2i\sqrt{2})$, $x=0$.

§ **58**, p. 131. **1.** $x^2+y^2-x-4=0$. **2.** $2x^2+2y^2+25x+24=0$.

3. $x^2+y^2-2x-9=0$, $9x^2+9y^2+62x-81=0$.

4. $5x^2+5y^2-50x+42y+9=0$.

5. $x^2+y^2-5x+4=0$, $4x^2+4y^2+65x+16=0$.

§ **59**, p. 132. **5.** $x^2+y^2=3y+14$.

§ **60**, p. 134. **1.** $(\frac{2}{13}, \frac{3}{13})$. **2.** $x^2+y^2=\frac{1}{2}x$. **3.** $x^2+y^2=\frac{k^2}{a}x$.

4. $x=3$. **5.** $x^2+y^2-16x+48=0$.

§ **63**, p. 138. **1.** $2x+5y=1$. **2.** $5y-2x=10$. **3.** $(4, 4)$.

4. $6x+5y=1$. **6.** $(-3, -\frac{1}{2})$.

Exercises XVIII. p. 138.

1. $m=\pm\frac{3}{4}$. **2.** $(3, 5)$. **3.** $4x-3y=-32$.

4. $(1, 3)$; $(-2, 0)$, $y=3$, $x+2=0$, $(-2, 3)$. **5.** $x+y=14$.

8. $ax=by$, $2ab/(a^2+b^2)^{\frac{1}{2}}$. **10.** $x^2+y^2-3x-5y+6=0$.

12. $x\cos\theta+y\sin\theta=a$. **13.** $2y=x+5$. **14.** $x^2+y^2=xx_1+yy_1$.

15. $(1, 1)$, $(\frac{7}{5}, \frac{1}{5})$. **16.** $x_1^2+y_1^2+\frac{2g}{a}x_1+\frac{2f}{a}y_1+\frac{c}{a}$.

19. $x^2+y^2=4x+1$. **21.** $-2(gl+fm-n)/(l^2+m^2)$.

24. $x^2+y^2=\frac{1}{2}y+4$. **27.** $x^2+y^2-\frac{9}{2}y+4=0$.

28. $(x^2+y^2+b)y_1=(x_1^2+y_1^2+b)y$; $x^2+y^2+2ky+b=0$, where
$k=\{-mn\pm\sqrt{(l^2+m^2)(n^2+bl^2)}\}/l^2$.

30. $(x-a)^2+(y-b)^2=r^2$. **31.** $2\sqrt{[-(x_1^2+y_1^2+2gx_1+2fy_1+c)]}$.

39. $\frac{3}{2}\sqrt{[-2(x_1^2+y_1^2+2gx_1+2fy_1+c)]}$, $\frac{m+n}{mn}\sqrt{[-mn(x_1^2+\text{etc.})]}$.

40. $2\sqrt{2}$, $-\sqrt{2}$. **42.** $4x=3y$. **43.** $x=1$, $y=1$.

§ **70**, p. 150. **1.** $y^2=4ax$. **2.** $y^2=4(x-1)$. **3.** $x^2=4ay$.
4. $16x^2-24xy+9y^2-70x-10y+100=0$.

§ **71**, p. 151. **1.** $5x^2+9y^2=45$.

Exercises XIX. p. 161.

1. $y=a$. **2.** $x+y=2$. **3.** $xy=1$.
4. $x+y=\frac{a}{2}$. **5.** $x+y=1$. **6.** $2x+3y=5$.
7. $(x-2)^2+y^2=a^2$. **8.** $(x-2)^2+(y-3)^2=0$. **9.** $4x^2+y^2=4$, 2π.
10. $x^2+y^2=4a^2$. **11.** $(3x-2)^2+9y^2=1$. **12.** $x^2+y^2-bx=0$.
13. $x^2=4a(y-a)$. **14.** $x^2+16=8y$. **15.** $3x^2-y^2+16x=64$.
16. $2xy=a^2$. **17.** $x^2+y^2=a^2$, $\frac{x^2}{4}+y^2=\frac{4a^2}{9}$, $x^2+\frac{y^2}{4}=\frac{4a^2}{9}$.
19. $7x^2+2xy+7y^2+6x-6y-9=0$. **21.** $(x-y)(k-2y)=ky$.
22. (i) $x^2=2ay$; (ii) $(x-y)^2=4ay$.
23. (i), (iii) $x^2+b^2=2(b\pm a)y+a^2$; (ii) $x^2=4ay$, $x=0$.
24. $(x+y)^2=2a(2x-a)$. **25.** $(x+y)^2=2a(2y-a)$.
26. $(x-y)^2=2a(2x-a)$. **27.** $x^2=2a(x+y-a)$.
28. $y^2=2a(y\pm x)$. **29.** $x^2=a(a\pm 2y)$.
30. $(x\pm y-a)^2(x^2+y^2)=a^2y^2$. **31.** $xy=1-x$.
43. (i) $x(a-x)^2=y^2(2a-x)$; (ii) $4y^2(3a-2x)=(2x-a)^3$.

Miscellaneous Examples I. p. 165.

4. $5, -\frac{1}{4}, \frac{1}{5}$. **6.** $1:1$. **7.** (a, d). **8.** $(1, 1)$.
9. $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$. **11.** $(9, 6)$, $(1, 8)$, $(-7, -4)$. **12.** $(1, 1)$.
13. $\left(\frac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_2+m_3}, \frac{m_1y_1+m_2y_2+m_3y_3}{m_1+m_2+m_3}\right)$. **14.** $\left(\frac{\Sigma mx}{\Sigma m}, \frac{\Sigma my}{\Sigma m}\right)$.
17. $4x-3y+1=0$. **18.** $x=-4+2t$, $y=-1+4t$, $2x-y+7=0$.
19. 2 ft. per sec., -1 ft. per sec.; $x+2y=3$.
20. (i) 0, 32, 64, 96, 128 ft. per sec.; (ii) $\frac{1}{2}$, 1, $1\frac{1}{2}$, $2\frac{2}{5}$ sec.;
(iii) 32 ft. per sec. per sec.; (iv) 16, 144, 80 ft.
21. (i) $v=32t$; (ii) $t=v/32$; (iii) $s=16t^2$; (iv) $v=8\sqrt{s}$.
23. (i) a; (ii) u; (iii) $-u/a$.
24. $x=a+(c-a)t/t_1$, $y=b+(d-b)t/t_1$; $(c-a)(y-b)=(d-b)(x-a)$.
25. $x=2+4t$, $y=1+3t$. **27.** $x-4y+14=0$, $\left(-\frac{6}{17}, \frac{58}{17}\right)$, $\frac{7}{17}\sqrt{17}$.
29. $83x-344y+107=0$, $\frac{28}{65}$. **30.** $\left(\frac{35}{8}, \frac{7}{8}\right)$. **31.** $\left(\frac{5}{4}, \frac{13}{4}\right)$.
32. $\frac{1}{31}\sqrt{986}$, $\frac{1}{17}\sqrt{986}$. **33.** $(-1, 7)$, $(-4, 3)$, $(4, -3)$, $(7, 1)$.
34. $(b+d-a, c+a+b)$, $(-b-d-a, -c-a+b)$. **35.** $(a, a+b)$, $(a+b, b)$.
36. $bx+(b+c)y=bc$, $(b+c)x+cy=bc$.

38. $\left(\left\{\frac{1}{2}(r+p)\mp\frac{1}{2}(s-q)\right\},\ \left\{\frac{1}{2}(s+q)\pm\frac{1}{2}(r-p)\right\}\right)$. **39.** $(a/2,\ 0)$.

40. $\frac{8}{4}$, $\left(\frac{13}{7},\ \frac{11}{7}\right)$. **41.** $\frac{8}{3}$, $\frac{32}{3}$, $\left(-\frac{3}{5},\ \frac{17}{5}\right)$.

43. $2x-3y+7=0$. **44.** $2y=x,\ 3y+x=0$.

45. A straight line. **46.** $y+x=7,\ 3y-5x+3=0$.

47. $acd/(bc+ad-ab),\ bcd/(bc+ad-ab)$. **48.** $3x-y+7=0$.

49. $(x-1)^2+(y+2)^2=36$. **51.** $u=1/(p+qt)$.

52. $(2,\ 4)$. **54.** $(3,\ 1+\sqrt{5})$.

55. $2y=9,\ 72x+154y+9=0,\ 3x=4y+9,\ 15x+8y=81$.

56. $y-b=\dfrac{\tan\alpha\pm m}{1\mp m\tan\alpha}(x-a)$,

$$(y-2)(13\pm3\sqrt{20})=(13\pm2\sqrt{20})(x-1),$$
$$(y-4)(13\pm3\sqrt{20})=(13\pm2\sqrt{20})(x-3).$$

57. $\left(-\dfrac{2lmg+l^2f+2nl-m^2f}{l^2+m^2},\ -\dfrac{2lmf+m^2g+2mn-l^2g}{l^2+m^2}\right)$.

Let the line pass through $(a,\ b)$; the locus is then

$$(x-a)^2+(y-b)^2=(f-a)^2+(g-b)^2.$$

61. (i) $\frac{700}{9}$; (ii) $\left(\frac{77}{18},\ \frac{20}{9}\right)$.

62. $4x+7y=65,\ 7x-4y=65$; $3(y-5)=(32\pm\sqrt{655})(x-4)$.

63. $x^2+y^2-2x-2y+1=0$. **65.** $(-5,\ 0),\ \left(0,\ \frac{5}{3}\right)$.

66. $(4,\ 12),\ (-3,\ 5)$. **67.** 2; $\frac{1}{2}$.

68. $\dfrac{OA}{3}\sqrt{3}$; $\left(0,\ \dfrac{OA}{3}\sqrt{3}\right)$. **69.** $\dfrac{(ad-bc)^2}{2bd}$. **70.** $\frac{8}{3}\sqrt{a^2+b^2}$.

73. The times are given by the roots of the equation

$$\Sigma(mn'-m'n)t^2+\Sigma(bn'-b'n+c'm-cm')t+\Sigma(bc'-b'c)=0.$$

75. $(x-y)(x+y-1)(x^2+y^2-x-y)=0$. This equation represents the diagonals and the circumcircle of the square $OACB$.

76. $y=4x$. **77.** $x^2=4(y-2)$. **78.** $x^2+2xy+y^2-8x-32y+130=0$.

79. $7x^2+16y^2=112$. **80.** $24xy+7y^2-48x-64y+64=0$.

81. $4x^2+4xy+y^2-26x-58y+166=0$.

82. $101x^2-48xy+81y^2+314x+1050y+1889=0$.

83. $13x^2-12xy-3y^2+80x-10y+25=0$.

84. (i) $x^2+y^2=a^2$; (iii) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$. **85.** $y=x$; $\frac{1}{2}(a+b)^2-4c$.

87. $hx+ky=r^2$. **88.** $(0,\ -3),\ (-2,\ -1)$.

89. $(1+2hm-m^2)(x^2+y^2)+2c(h-m)x-2c(1+hm)y=0$.

90. (i) $ax^2+2hxy+by^2-\dfrac{2}{n}(gx+fy)(lx+my)+\dfrac{c}{n^2}(lx+my)^2=0$;

(ii) $a(x-p)^2+2h(x-p)(y-q)+b(y-q)^2$

$$-2\{(x-p)(ap+hq+g)+(y-q)(hp+bq+f)\}$$
$$\times\{l(x-p)+m(y-q)\}/(lp+mq+n)$$
$$+(ap^2+2hpq+bq^2+2gp+2fq+c)\{l(x-p)+m(y-q)\}^2/(lp+mq+n)^2.$$

92. $x^2+y^2=a(x+y),\ x+y=0$. **93.** $-\frac{1}{5}$, $\frac{5}{2}\sqrt{20}$, $\left(-\frac{23}{9},\ \frac{15}{9}\right)$.

96. $(x-a)^2+(y-b)^2=a^2+b^2$.

98. $(g\cos\alpha+f\sin\alpha+p)^2=g^2+f^2-c$;
$(gx+fy+x^2+y^2)^2=(g^2+f^2-c)(x^2+y^2)$.

100. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Miscellaneous Examples II. p. 174.

1. $y=\frac{5}{12}x$, $x=0$.

2. $x^2+y^2+2x-12y+17=0$, $9x^2+9y^2-62x-28y+73=0$.

3. $a_1(y-b_3)(b_2x-a_2y)-b_1(x-a_3)(b_3x-a_3y)+(b_3x-a_3y)(b_2x-a_2y)=0$.

5. $(y-y_2)(x_1y-y_1x)=\lambda(y-y_1)(x_2y-y_2x)$ if $OM:ON=\lambda$.

6. $9x+2y=9$, $x+3y=6$, $11x+6=7y$, $(\frac{3}{5}, \frac{9}{5})$. **7.** See Ex. 9.

8. $(5, \frac{10}{3})$, $(3, 4)$. **10.** $\xi(A\sqrt{3}+B)+\eta(A+B\sqrt{3})+2C=0$.

12. $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$.

16. $\sqrt{(h^2-ab)/(am^2-2hlm+bl^2)}$. **17.** $\frac{1}{2}c^2(m_2-m_1)/(a+bm_1)(a+bm_2)$.

18. See § 38. **20.** $(1, l+m+n+lmn)$.

27. $\left(\dfrac{b-\beta}{\alpha b-a\beta}, \dfrac{a-\alpha}{\alpha b-a\beta}\right)$. **31.** $\frac{25}{2}$.

35. $x-y=\frac{1}{2}(a+b)$, $(b+c)x-by=\frac{1}{2}(a+b)(b+2c)$,
$(c+a)x-ay=\frac{1}{2}(a+b)(a+2c)$;
$x^2+y^2-2(a+b)x-(a+b)y+a^2+3ab+b^2=0$.

37. $(a^2+b^2)(x^2+y^2)+2c(ax+by)=0$. **42.** $(al+bm+n)^2=r^2(l^2+m^2)$.

44. $(2+\sqrt{5})x-y=\sqrt{5}+1$. **47.** d/b. **51.** $y=x+xy$.

53. Let the circle be $x^2+y^2=a^2$, etc.; the locus is
(i) $y^2=(x+2a)^2(a-x)/(x+3a)$, or (ii) $y^2=x^2(x-a)/(a+3x)$ according as the points move in the same direction or in opposite directions.

55. $(x^2+y^2-2ax)^2=b^2(x^2+y^2)$.

56. $(bx-ay)(ay+bx-ab)=2c^2(x-a)(y-b)$.

57. $bb'x(a-x)=aa'(b-y)y$. When AB and $A'B'$ are parallel this becomes $(bx-ay)(bx+ay-ab)=0$, or the line AB and the perpendicular through O to AB.

58. $2a(x^2+y^2+2c^2)=(4c^2+a^2)(x+y)$. **59.** $x^2y^2+a^2x^2-a^2y^2=0$.

60. $x(\eta-b)\eta a+y(\xi-a)\xi b+ab\xi\eta=0$.

61. Let A be $(0, 0)$; let the circle be $(x-a)^2+y^2=r^2$. Then the locus is
$2ax=a^2-r^2$.

64. $y^3+ay^2+x^2y-ax^2=0$. **70.** When $c=b$, the locus is a cissoid.

Exercises XX. p. 187.

5. $y-1=x^3$, $y+1=x^3$, $y+b=x^3$.

§ 80, p. 190. **4.** (i) 2·87, −0·87; (ii) 3, −1; (iii) 2·58, −0·58;
(iv) 2·87, −0·87; (v) 2·91, −0·91; (vi) 2·95, −0·95.

6. (i) $(0, -1)$, $(3, 2)$; (ii) $(2·42, 0·03)$, $(-0·76, 1·09)$; (iii) $(2, -1)$ *twice*; (iv) the roots are imaginary; the straight line does not meet the curve at all.

7. (i) 2·33, −0·20, −2·13; (ii) 0·72; (iii) 2·11, −0·68; (iv) 1·00, 0·455, −2·06; (v) 1·16; (vi) 1·86, −0·37.

8. (i) (−0·86, 0·74), (1·40, 1·96); (ii) (3·53, −3·76), (5·41, 4·05).

9. (i) −0·86, 1·40; (ii) 3·53, 5·41.

Exercises XXIII. p. 213.

10. (i) $\frac{3}{2}$, $\frac{4}{3}$; $-\frac{3}{2}$, $-\frac{4}{3}$; $(+\infty, +\infty; -\infty, -\infty)$; (ii) $\frac{5}{2}$, $-\frac{7}{2}$; (iii) 1·30, 0·30; −2·30, −3·30; 1·32, 0·36; −0·57, 4·14; (iv) −0·69, 0·41; 0·63, −1·69; (v) −2·18, 0·52; 1, −1.

§ 89, p. 217. 5. $(x\sin\alpha - y\cos\alpha) = gx^2/2V^2\cos\alpha$.

Exercises XXIV. p. 217.

1. $(y-x+2)^2 = 9y$.

2. $4(y+x) = (y-x)^3$, or, turning axes through 45°, $\eta = \frac{1}{2}\xi^3$.

4. $(x+1)^2y^2 = x(2y+1)^2$.

5. $x^3 = (y+x)^2$.

6. $x(y+1)(xy-1) = y^3(x+1)^2$.

7. $x^3+y^3 = 3xy$.

8. $y = (x+1)(x-1)^2/(3x^2+1)$.

Exercises XXV. p. 227.

2. $y+x=0$, $y=3x-2$.

3. (i) $y=x-1$; (ii) $5x=y+9$.

4. (i) $x=3y+2$; (ii) $8x+3y+2=0$; (iii) $2x+3y+2=0$, $4x=3y+2$.

5. (i) $3y=x+1$; (ii) $3y=10x+1$; $\frac{1}{3}(1\pm\sqrt{10})$, maximum and minimum values of y.

6. (i) $-\frac{1}{2}$, -2; (ii) $\pm 2\sqrt{5}$.

10. (i) (0, 0), (3, 1), (0, 1) twice; (ii) (0, 0), (2, 0), (2, 3), $(1, -\frac{1}{2})$; (iii) (0, 0). $\left(a\left\{1-\left(\frac{b}{a}\right)^{\frac{2}{3}}\right\},\ b\left\{1-\left(\frac{a}{b}\right)^{\frac{2}{3}}\right\}\right)$.

§ 95, p. 231. 8. (i) $a+ve$, $ac>b^2$; (ii) $a-ve$, $ac>b^2$.

§ 96, p. 233. 2. (i) $y=3x-4$; (ii) $y+x=2$, $y=x-3$; (iii) $y=3x-7$, $y+5x=1$; (iv) $y=5x-7$; (v) $y=2(x-1)$, $y=2(x-3)$.

§ 99, p. 238. 1. (i) $y=-x-x^3\ldots$; (ii) $y=-x^2-x^3\ldots$; (iii) $y=-\frac{1}{6}+\frac{x}{36}+\frac{11x^2}{216}\ldots$; (iv) $y=1-2x+2x^2\ldots$; (v) $y=-1+3x-3x^2\ldots$; (vi) $y=-2+x-\frac{3}{2}x^2\ldots$; (vii) $y=2+2x+2x^2\ldots$

2. $y+x=5$.

Exercises XXVI. p. 244.

4. $5x+y=1$, $3x+y=2$, $y=x-2$, $y=3x-7$. **9.** $2y=x+3$.

10. (i) $y=x$, $\eta=\xi-\frac{\xi^3}{2}\ldots$; (ii) $y+7x=27$, $\eta=-7\xi+9\xi^3\ldots$;
(iii) $9y=20x-28$, $\eta=\frac{20}{9}\xi+\frac{25}{27}\xi^2$; (iv) $y=3$, $\eta=-2\xi^2\ldots$;
(v) $y=1$, $\eta=7\xi^2\ldots$.

16. $y=x-1$. **17.** $(\frac{1}{2}, \frac{23}{2})$.

19. $3y=x-3$. **20.** $h=1$, $k=2$, or $h=\frac{5}{2}$, $k=1$, or $h=-\frac{1}{2}$, $k=3$.

§ 103, p. 252. **4.** (i) $y=x-10$, $y=13x-34$; (ii) $y=2(x-p)$, $2py=2hx-h^2$; (iii) $y=11x-6$, $y=2x-2$, $y+x=2$, $y=2x-6$, $y=11x-38$, $y=26x+2$, $y=47x+34$; (iv) $y=0$, $y=(b-a)^2(x-b)$; (v) $y=0$, $(n>1)$.

Exercises XXVII. p. 257.

[Abscissae only are given: first, turning points, then inflexions, then intercepts on x-axis.]

1. $\frac{6}{7}$; none; 2·33, −0·61. **2.** $\frac{3}{2}$; none; 4·62, −1·62.

3. 0, $\frac{4}{3}$; $\frac{2}{3}$; 0, 2. **4.** 1, −2; $-\frac{1}{2}$; 0·65, 1·33, −3·47.

5. 1, −1; 0; 0·35, 1·53, −1·88. **6.** 1·53, −1·53; 0; 2·40, 0·44, −2·84.

7. 2, 4·12, −0·12; 3·22, 0·78; 2 twice, 5, −1.

8. 0; $\frac{1}{3}$, 1; 2·08, −0·83. **9.** 1, 3·24, −1·24; 2·29, −0·29; none.

10. 3, 1; 0, 2·37, 0·63; 3·51, 2·10, −0·97.

13. (9) None. (10) (1, 2); (−1, −2). (11) None.
(12) $(\sqrt{2}, 3+2\sqrt{2})$; $(-\sqrt{2}, 3-2\sqrt{2})$. (13) (−1, 0); (−3, 8).
(14) $(\sqrt[3]{2}, 3\sqrt[3]{2}/2)$. (15) $(-\sqrt[3]{2}, -3\sqrt[3]{2}/2)$. (16) $(\sqrt[3]{4}/2, 3\sqrt[3]{2}/2)$.
(17) None. (18) (1, 2); (−1, 2). (19) None.

Exercises XXVIII. p. 267.

1. −2.

3. (The value of x is given second.) (i) $-\frac{1}{4}$, $\frac{1}{2}$. (ii) 3, 1; −1, −1. (iii) 5·83, 3·41; 0·17, 0·59. (iv) 3, 1; $\frac{1}{3}$, −1. (v) −0·15, 1·37; 2·15, 0·37. (vi) −0·46, 1·37; 0·46, −0·37. (vii) −13·03, 3·37; −0·07, 1·03.

4. (i) 0, $-\frac{32}{27}$; (ii) −1·38; (iii) 0, 2·09. **5.** $\eta=-c^2\xi^2$, $\eta=c^2\xi^2$.

6. 0, max.; −4, min. **7.** New origin is (2, 0); (2, 0); ±0·385.

8. $\frac{8}{3}$, 0; $y=-\frac{256}{27}$, (min.), 0 (max.). **9.** $10y-3-3y^2$; 3, $\frac{1}{3}$.

10. $32(y-9)(y-5)$; 9 min., 5 max.

11. $\frac{ah}{4}$. **12.** $\frac{l}{4}\times\frac{l}{2}$; $\frac{l}{6}\times\frac{l}{2}$. **13.** $\pi:4$.

14. (i) $\sqrt{ab}$; (ii) $\sqrt{ab}$; (iii) $\sqrt{ab}$; (iv) $\sqrt{ab}$; (v) b; (vi) a.

15. (i) $2ab$; (ii) $b+a+2\sqrt{ab}$; (iii) $2ab$; (iv) 1/4. **16.** 3/2.

20. $\frac{3x-1}{x-1}>1$ if $x>1$, or $x<0$; $\frac{3x-1}{x-1}=1$ if $x=0$; $\frac{3x-1}{x-1}<1$ if $0<x<$

21. $2+2\sqrt{2}$ min., $2-2\sqrt{2}$ max. **22.** (1, 1) max., (2, 2) max.

23. (i) $\frac{1}{3}(7+\sqrt{13})$ max., $\frac{1}{3}(7-\sqrt{13})$ min.; (ii) 2 min., $\frac{10}{9}$ max.;
(iii) $-4-2\sqrt{6}$ max., $-4+2\sqrt{6}$ min.;
(iv) $\frac{1}{2}(9+2\sqrt{14})$ min., $\frac{1}{2}(9-2\sqrt{14})$ max.; (v) 0 max., -1 min.;
(vi) $2+\sqrt{3}$ max., $2-\sqrt{3}$ min.

§ 110, p. 273. (i) 2·7321, –0·7321; (ii) –0·8772, 0·1620; (iii) 1·8431, 0·4(

§ 111, p. 275. **2.** The root lies between (i) 1 and 2; (ii) 2 and 3; (iii) –1 and –2; (iv) 0 and 1; (v) 1 and 2.

3. The roots lie between (i) 2 and 3; (ii) $-\frac{1}{2}$ and 0, 0 and $\frac{1}{2}$, $\frac{1}{2}$ and 1 (iii) –1 and 0, 0 and 1, 3 and 4; (iv) –2 and –1, 1 and 2.

Exercises XXIX, p. 279.

1. 2·095.	**2.** 1·213.	**3.** 0·466.	**4.** 1·552.
5. 1·276.	**6.** 0·755.	**7.** 1·220, –1·566.	**8.** 1·420, –0·(
9. 2·597.	**10.** 2·858.	**11.** 0·226.	**12.** 0·347.

Exercises XXX, p. 288.

1. (i) None; (ii) two. **2.** (i) None; (ii) three.

3. (i) One, $(\frac{1}{3}, \frac{4}{3})$; (ii) two. **4.** (i) None; (ii) two.

5. (*a*) (i) None; (ii) two. (*b*) (i) None; (ii) two.

6. (*a*) (i) None; (ii) three. (*b*) (i) None; (ii) three.

7. (*a*) (i) None; (ii) two. (*b*) (i) None; (ii) two. **9.** $y=x$

10. (i) $y=0$, $y=x$; (ii) $y=0$, $y=x$; (iii) $y=0$, $y=x-1$; (iv) $y=0$, $y=x$

12. (i) $y=x+3$, $x=2$; (ii) $y=x+2$, $x=2$; (iii) $y=1$; (iv) $y=1$; (v) y $x=1$, $x=2$; (vi) $y=1$, $x=3$, $x=4$.

13. (3, 2).

Exercises XXXI, p. 294.

1. (i) $y=x$, $y+x=0$; (ii) $y=x$, $y+x=0$; (iii) $x=0$, $y=x$;
(iv) $y=x$, $x+y+1=0$; (v) $y=2x$, $x=2y$; (vi) $y=2x+1$, $y=x-2$;
(vii) $y=0$, $y=x$, $y=2x$; (viii) $x=0$, $y=0$, $y=x$, $y=2x$;
(ix) $y+1=0$, $x+1=0$, $x+y=2$; (x) $8\sqrt{2}x+8y+3=0$, $8\sqrt{2}x-8y-3$

16. $5(x+y)=1$. The curve appears above this line to the right, belo on the left, and cuts it at (·075..., ·125...).

18. $(2x+y-1)(x-y+1)$.

23. (i) $x=1$, $y=0$; (ii) $x+y+1=0$; (iii) $x-3y=1$.

§ 122, p. 299. **3.** (i) $-x/y$; (ii) $(x+2y)/(y-2x)$; (iii) $(a+bx)/y$;
(iv) $-(ax+hy+g)/(hx+by+f)$; (v) $(2ax-y^2-3x^2)/2y(a+x)$;
(vi) $(y^2+2xy-2x)/(2y-2xy-x^2)$;
(vii) $(2axy-xy^2-x^3)/(x^2y+y^3-ax^2)$; (viii) $(4xy-4x^3+5x^4)/2(y-$

4. (i) $3/(x+2)^2$; (ii) $-2/x^3$; (iii) $-n/x^{n+1}$; (iv) $(1-2x^3)/(1+x^3)^2$; (v) $x(2-x)/(1-x+x^2)^2$; (vi) $1/2(1+x)\sqrt{x+x^2}$; (vii) $x(2x+3)/(2x+2)\sqrt{x+x^2}$.

6. (i) $2(1-x^2)/(1-x+x^2)^2$;
(ii) $\{(aB-bA)x^2+2(aC-cA)x+(bC-cB)\}/(Ax^2+Bx+C)^2$;
(iii) $x/\sqrt{x^2+1}$; (iv) $-x/(x^2+1)^{\frac{3}{2}}$; (v) $\{x+\sqrt{x^2+1}\}/\sqrt{x^2+1}$;
(vi) $(3x+2b+a)/2(x+b)^{\frac{1}{2}}$.

Exercises XXXIII. p. 315.

1. $16x^2+24xy+9y^2+30x-48y-36=0$; $4x+3y=0$; $\frac{12}{5}$; $(\frac{9}{25}, -\frac{12}{25})$.

2. (i) $x^2+2xy+y^2-12x-16y+50=0$; $x+y=7$; $\sqrt{2}$; $(\frac{13}{4}, \frac{15}{4})$.
(ii) $9x^2+24xy+16y^2-52x-86y+89=0$; $3x+4y=10$; $\frac{2}{5}$; $(\frac{52}{25}, \frac{47}{50})$.
(iii) $16x^2-24xy+9y^2-2x+14y-14=0$; $4x-3y=1$; $\frac{2}{5}$; $(\frac{53}{50}, \frac{27}{25})$.
(iv) $9x^2-24xy+16y^2-2x+86y+14=0$; $3x-4y=7$; 2; $(\frac{7}{5}, -\frac{7}{10})$.
(v) $144x^2-120xy+25y^2+130x+312y-169=0$; $12x-5y=0$; 2; $(\frac{5}{26}, \frac{6}{13})$.

3. The equation represents the locus of a point which moves so that its distance from the point $(0, \frac{1}{2})$ is equal to its distance from the line $x=\frac{1}{2}$.

4. (i) $(0, \frac{1}{4})$, $y+\frac{1}{4}=0$; (ii) $(0, -\frac{1}{4})$, $y=\frac{1}{4}$; (iii) $(\frac{1}{4}, 0)$, $x+\frac{1}{4}=0$; (iv) $(-\frac{1}{4}, 0)$, $x=\frac{1}{4}$.

5. $\eta=\xi^2$; 1; $(2, -\frac{11}{4})$; $4y+13=0$.

10. $y=2x^2-7x+2$.

12. (i), (iv) upwards; (ii), (iii) downwards.

13. (i) a; (ii) $a-b-c+d$.

14. $x^2=4ay$; $4a$; $(0, a)$; $y+a=0$.

15. Let the point be $(0, a)$, the line $y=0$. Then locus is $x^2=4ay$; focus, $(0, a)$; vertex, $(0, 0)$; directrix, $y+a=0$.

24. $y^2=2a(x-4a)$; $2a$; $(4a, 0)$; $\left(\frac{9a}{2}, 0\right)$.

28. $(2a-c, 2a)$; $y=2a$.

Exercises XXXIV. p. 319.

1. $e=\frac{5}{6}$; $SX=\frac{11}{5}$; $OX=\frac{36}{5}$.

2. $e=\frac{6}{7}$; $SX=\frac{13}{7}$; $AX=1$.

3. $e=\frac{4}{5}$; $SX=\frac{9}{4}$.

4. $OB=3\sqrt{3}$; $SX=9$; $SA=3$.

5. $OA=2\sqrt{6}$; $OB=2\sqrt{2}$; $e=\frac{1}{3}\sqrt{6}$.

6. $b=3$; $OS=4$; $OX=\frac{25}{4}$; $AX=\frac{5}{4}$.

7. $a=5$; $OS=3$; $SA=2$; $SX=\frac{16}{3}$.

8. $\frac{18}{5}$.

9. $OA=2$ in., $OB=\sqrt{3}$ in.

11. $e=\frac{1}{50}$.

12. $e=\frac{1}{2}\sqrt{2}$.

16. The parallel through A to the directrix.

Exercises XXXV. p. 322.

1. $\frac{3}{5}$. 2. $e=\frac{4}{5}$; $\frac{18}{5}$. 3. $\frac{x^2}{81}+\frac{y^2}{9}=1$.

4. $\frac{7}{3}$. 5. (3·2, 2·8) is inside, others outside.

6. The four points $\left(\pm\frac{ab}{\sqrt{a^2+b^2}},\ \pm\frac{ab}{\sqrt{a^2+b^2}}\right)$; $\frac{2\sqrt{2}\,.\,ab}{\sqrt{a^2+b^2}}$.

7. $\frac{17}{5}$. 9. $\sqrt{\frac{6}{7}}$. 12. $\frac{(2x-y+1)^2}{45}+\frac{(x+2y-3)^2}{125}=1$.

13. $\frac{16(3x-4y+1)^2}{2425}+\frac{16(4x+3y+2)^2}{3721}=1$.

14. (i) $(0, 0)$; $x=0$, $y=0$; $2\sqrt{7}$, $2\sqrt{5}$;
 (ii) $(0, 0)$; $y=0$, $x=0$; $2(a+b)$, $2(a-b)$.
 (iii) $(1, -2)$; $y+2=0$; $x=1$; 4, $2\sqrt{3}$. (iv) $(0, 0)$; $x+y=0$, $x=y$; 6, 4
 (v) $(0, 0)$; $4x=3y$, $3x+4y=0$; $2\sqrt{5}$, $8\sqrt{5}/5$.
 (vi) $(-3/5, 1/5)$; $2x+y+1=0$, $x-2y+1=0$; $2\sqrt{6}$, $2\sqrt{35}/5$.
 (vii) $(1, -1)$; $y+1=0$, $x=1$; 4, $2\sqrt{3}$.
 (viii) $(1/2, -2/3)$; $3y+2=0$, $2x=1$; $\sqrt{5}$, $2\sqrt{2/3}$.
 (ix) $(2/3, -1)$; $3x=2$, $y+1=0$; $2\sqrt{3}$, 2
 (x) $(-2/3, -1)$; $3x+2=0$, $y+1=0$; $2\sqrt{3}$, 2.

15. $17x^2+16xy+17y^2+80x-80y-200=0$; $4\sqrt{2}$.

16. $91x^2+24xy+84y^2-200x-320y+400=0$; $\left(\frac{9}{5}, \frac{28}{15}\right)$.

22. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; πab. 23. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

28. If $CA=a$, $CB=b$, the points are

$$\left(\frac{ab^2}{a^2-b^2},\ \pm\frac{ab\sqrt{(a^2-2b^2)}}{a^2-b^2}\right),\ \left(\pm\frac{ab\sqrt{(b^2-2a^2)}}{b^2-a^2},\ \frac{ba^2}{b^2-a^2}\right).$$

It will be noticed that one pair of points, at least, is imaginary.

30. $\left(\frac{3}{5}, \frac{2}{5}\right)$. 31. $ax+bmy=0$.

32. $7x^2-2xy+7y^2-14x-30y+39=0$; $7x-y=7$. 33. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{x^2+y^2}{a^2+b^2}$

Exercises XXXVI. p. 325.

1. (i) $\frac{4}{5}$; (ii) 10; (iii) 6. 2. 14; $2\sqrt{33}$.

5. $15x^2+16y^2-300x+1200=0$; $e=\frac{1}{4}$. 6. $e=\frac{a}{b}$.

Exercises XXXVII. p. 328.

1. $e=\frac{6}{5}$; $SX=\frac{11}{6}$; $CX=\frac{25}{6}$. 2. $e=\frac{7}{6}$; $SX=\frac{13}{7}$; $AX=\frac{6}{7}$.

3. $e=\frac{5}{4}$; $SX=\frac{9}{5}$. 4. $e=\frac{5}{3}$; $SX=\frac{16}{5}$.

5. $CB=6\sqrt{3}$; $SX=9$; $SA=6$. 6. $CA=2\sqrt{6}$; $CB=2\sqrt{3}$; $e=\frac{1}{2}\sqrt{6}$

7. $b=\frac{15}{4}$; $CS=\frac{25}{4}$; $CX=4$; $AX=1$.

8. $a=3$; $CS=5$; $SA=2$; $SX=\frac{16}{5}$.

9. $\frac{50}{3}$. 10. $a=2$ in.; $b=2\sqrt{3}$ in.

Exercises XXXVIII. p. 330.

1. $\frac{5}{3}$. **2.** $\frac{5}{4}$; $\frac{9}{4}$. **3.** $\frac{x^2}{81}-\frac{y^2}{9}=1$.

4. $\frac{25}{13}$. **5.** $(-5, 2)$, $(-7, -4)$ are inside.

6. $\left(\frac{\pm ab}{\sqrt{(b^2-m^2a^2)}}, \frac{\pm mab}{\sqrt{(b^2-m^2a^2)}}\right)$, $\left(\frac{\pm ab}{\sqrt{(b^2-m^2a^2)}}, \frac{\mp mab}{\sqrt{(b^2-m^2a^2)}}\right)$.

When $m=\frac{b}{a}$ we get the asymptotes. **9.** $\frac{3}{2}\sqrt{21}$.

10. r is real and finite, infinite, or imaginary, according as

$$\cos^2\theta > \frac{1}{e^2}, \quad = \frac{1}{e^2} \quad \text{or} \quad < \frac{1}{e^2}.$$

12. (i) $y+2=0$, $x=1$; 6, 4; $(1, -2)$. (ii) $x=0$, $y=0$; $2b$, $2a$; $(0, 0)$.
(iii) $y+1=0$, $x=1$; 4, $2\sqrt{3}$; $(1, -1)$.
(iv) $3y+2=0$, $2x=1$; $\sqrt{5}$, $2\sqrt{2/3}$; $(1/2, -2/3)$.
(v) $2x+y=0$, $x=2y$; $4\sqrt{2}$, $2\sqrt{3}$; $(0, 0)$.
(vi) $x+y=2$, $y=x+1$; $4\sqrt{2}$, $3\sqrt{2}$; $(1/2, 3/2)$.

16. $x^2+4xy+y^2-12x-6y+3=0$; $6\sqrt{2}$. **17.** $2xy-6x-6y+9=0$; $(3, 3)$.

Exercises XXXIX. p. 333.

1. (i) $5/4$; (ii) 8 in.; (iii) 6 in. **2.** 10.

4. The locus is likewise a hyperbola whose foci are the centres of the given circles.

5. $220x^2-36y^2=495$ $(e=\frac{8}{3})$ and $252x^2-4y^2=63$ $(e=8)$.

Exercises XL. p. 338.

2. (i) $2x-3y=0$, $2x+3y=0$; (ii) $x+y=0$, $x-y=0$;
(iii) $3x-2y=5$, $3x+2y=1$; (iv) $2x+y=0$, $2x-11y=0$.
(v) $\frac{x}{a}-\frac{y}{b}=0$, $\frac{x}{a}+\frac{y}{b}=0$.

9. $x=3$, $y+2=0$; $(3, -2)$. **10.** $ax+c=0$, $ay+b=0$.

11. (i) $\sqrt{2}$; (ii) $\frac{1}{3}\sqrt{3}$. **12.** 0, $\frac{3}{2}$.

13. (i) $2x-y+1=0$, $x+y-2=0$; $(\frac{1}{3}, \frac{5}{3})$.
(ii) $3x+2y+1=0$, $x-2y-2=0$; $(\frac{1}{4}, -\frac{7}{8})$.

14. $(2x+3y-8)(x-2y+3)=22$. **18.** $y=\frac{2c}{t}-\frac{x}{t^2}$.

22. The asymptotes being the axes, the constant length is the algebraic difference of the ordinates or abscissae of the fixed points.

26. Let the given asymptote be AOB, the given tangent CED, with point of contact E, the other tangent BL. Let AOB and CED intersect at C, BL and AOB in B. Cut off $ED=CE$. Join BD. Let CL, parallel to BD, meet BL in L. LD is the second asymptote.

28. $xy=\frac{ab}{4}$; $\frac{1}{2}\sqrt{2ab}$.

Exercises XLI. p. 346.

2. $-36^\circ\ 52'$.

4. $\left(\frac{\sqrt{3}}{2}a, \frac{b}{2}\right)$; $\left(\frac{\sqrt{2}}{2}a, \frac{\sqrt{2}}{2}b\right)$; $\left(\frac{1}{2}a, \frac{\sqrt{3}}{2}b\right)$.

8. $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$.

9. $\frac{ma}{b}$.

10. $\frac{ma}{b}$.

18. $a^2l^2 + b^2m^2 = 1$.

22. $\tan^{-1}\left(\frac{b^2}{a^2}\cot\theta\right)$.

26. $n^2(a^2m^2 + b^2l^2) = l^2m^2(a^2 - b^2)^2$.

30. (i) $\sqrt{(ab)}$; (ii) $a - b$; (iii) $x\sqrt{a^2+ab} + y\sqrt{b^2+ab} = a^2 - b^2$.

(iv) $\left(\frac{a^2(a^2-ab-b^2)}{(a^2-ab+b^2)(a^2+ab)^{\frac{1}{2}}}, \frac{b^2(b^2-ab-a^2)}{(a^2-ab+b^2)(b^2+ab)^{\frac{1}{2}}}\right)$.

33. $\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} = 1$.

38. $x^2 + y^2 = (a+b)^2$.

Exercises XLII. p. 355.

1. $-\frac{b^2}{a^2}$.

2. $50y = 63x$.

6. $3y = 5x$.

11. $\left(\pm\frac{a}{2}\sqrt{2}, \pm\frac{b}{2}\sqrt{2}\right)$.

12. See § 130, Ex. 3.

13. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{2}$; $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$.

14. $(2, 2)$; $x + y - 4 = \pm\frac{1}{3}(x - y)$.

16. $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$.

17. $\left(\frac{a^2mc}{b^2+a^2m^2}, \frac{b^2c}{b^2+a^2m^2}\right)$.

Exercises XLIII. p. 359.

1. Let SZ, $\perp^r$ SP, meet the directrix in Z. ZP is the tangent at P.

3. Let Z be the point. Join ZS, and let the $\perp^r$ to SZ at S cut the conic in P, P'; ZP, ZP' are the tangents required.

15. Let the $\odot^{\text{les}}$, centre S, radius $SU = eKL$, meet the $\odot^{\text{le}}$ on SK as diameter in U, U'. Let SU, SU' cut the conic in P, P'. KP, KP' are the required tangents.

17. Let S be focus, PT tangent at P, Q a point on the conic. Let SZ, $\perp^r$ SP, meet PT in Z. Let SZ', the external bisector of $P\hat{S}Q$, meet PQ in Z'. ZZ' is the directrix. Whence the vertex is easily found.

Exercises XLIV. p. 363.

3. Draw SX $\perp^r$ the directrix, mid point A. Let AY $\perp^r$ SX meet $\odot^{\text{le}}$ on OS as diameter in Y (and Y'). Let SY meet dirx. in M. Let MP $\perp^r$ dirx. meet OY in P. P is point required.

4. Let M be the image of S in PT; MX, drawn perpendicular to MP, is the directrix.

5. Produce $SY \perp^r PT$ to M so that $YM = SY$. With centre Q, radius SQ, describe a ⊙^le^. Draw MM' (and MM''), the tangent to the ⊙^le^. Draw $MP \perp^r MM'$ to meet PT in P. P is the point required and MM' is the directrix.

6. Draw $SY \perp^r PT$ and produce to M so that $SY = YM$. Draw $MX \perp^r$ axis. MX is the directrix. Let $MP \perp^r MX$ meet PT in P. P is the point required.

16. Draw $SM \perp^r$ given line to meet dirx. in M. Bisect SM in Y. $YP \perp^r SM$ is the tangent required.

17. $PT = 6\sqrt{10}$; $TN = 18$.

19. t is the reciprocal of the gradient at the point t.

34. The join of the feet of the $\perp^s$ from the focus on the tangents is tangent at vertex. Whence the required construction.

40. Let the tangents be PQ, QR, RP. Let QR touch the parabola at T. The circle which touches PQ at Q and passes through T will cut the circumcircle of the triangle PQR again at the focus S. If M and N are the images of S in two of the tangents, MN is the directrix.

44. $y^2 = -x^3/(a+x)$, a cissoid.

Exercises XLV. p. 372.

2. Let the tangent ZPT meet dirx. in Z. Draw $SP \perp^r SZ$ to meet ZPT in P. Let PS', making $\angle S'PT = \angle SPZ$, meet $SX \perp^r$ dirx. in S'. C is mid point of SS'.

29. $(1-e^2)x$. This becomes l, where l is the semi-latus rectum of the parabola.

30. $(a^2-x^2)/x$.

36. Circles centres the foci, radii equal to major axis.

37. Circle centre second focus; radius equals the difference of the major axes.

38. Let join of P, Q, the given points, meet the asymptote in R. Produce PQ to T so that $QT = RP$. Draw $TC \perp^r$ the asymptote. C is the centre.

Exercises XLVI. p. 378.

4. $2a(t^2+1)^{\frac{3}{2}}$; $27ay^2 = 4(x-2a)^3$. 6. $t = \pm\sqrt{2}$. 7. $m^2 = (a-b)/c$.

15. $x = k(t^2-1)$, $y = 2k(t-1)$. 18. $y = \tan\theta\{x - 2a - a\tan^2\theta\}$.

19. $y = mx + l/2m$, where $2a^2m^2 + b^2 = +\sqrt{a^2l^2+b^4}$.

28. $ln + a(l^2+m^2) = 0$. 36. $(a, 2a)$, $(\frac{9}{4}a, 3a)$, $(\frac{25}{4}a, -5a)$.

Exercises XLVII. p. 385.

9. $$\left\{\frac{a^2-b^2}{a}\cos\theta\cos\phi\cos\tfrac{1}{2}(\phi+\theta)/\cos\tfrac{1}{2}(\phi-\theta),\right.$$
$$\left.\frac{b^2-a^2}{b}\sin\theta\sin\phi\sin\tfrac{1}{2}(\phi+\theta)/\cos\tfrac{1}{2}(\phi-\theta)\right\};$$
$$\left(\frac{a^2-b^2}{a}\cos^3\theta,\ \frac{b^2-a^2}{b}\sin^3\theta\right);\quad (a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{3}{2}}/ab.$$

24. $(a^2+b^2m^2)(y-mx)^2 = (a^2-b^2)^2m^2$.

Exercises XLVIII. p. 398.

2. (i) $(a^2l,\ b^2m)$; (ii) $(a^2l,\ -b^2m)$; (iii) $(2mc^2,\ 2lc^2)$; (iv) $\left(-\frac{1}{l},\ -\frac{2am}{l}\right)$.

3. (i) $a^2ll'+b^2mm'=1$; (ii) $a^2ll'-b^2mm'=1$; (iii) $2c^2(lm'+l'm)=1$; (iv) $l+l'+2amm'=0$.

4. (i) $(y-y_1)y_1=2a(x-x_1)$; (ii) $(x-x_1)\frac{x_1}{a^2}-(y-y_1)\frac{y_1}{b^2}=0$.

44. $\frac{x^2}{a^2}+\frac{2y^2}{a^2+b^2}=1$.

45. $\tan^2\phi(x^2+y^2-a^2-b^2)^2=4a^2b^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$.

46. (i) $x^2+y^2=a^2+b^2$; (ii) $x^2+y^2=a^2-b^2$.

Exercises XLIX. p. 405.

1. $\frac{x\xi}{a^2}+\frac{y\eta}{b^2}=\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}$. **2.** $\frac{x\xi}{a^2}-\frac{y\eta}{b^2}=\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}$.

4. (i) $xx_1/a^2+yy_1/b^2=1$; (ii) $xx_1/a^2-yy_1/b^2=1$; (iii) $axx_1+h(xy_1+x_1y)+byy_1+g(x+x_1)+f(y+y_1)+c=0$.

7. $2c(1+m^2)^{\frac{1}{2}}(h^2-ab)^{\frac{1}{2}}/(a+2hm+bm^2)$. **8.** $(h^2-ab)^{\frac{1}{2}}/(am^2-2hlm+bl^2)$.

9. $(x^2+y^2)(ax+by+c)^2=d^2(ax+by)^2$. **12.** $\frac{x}{\alpha}+\frac{y}{\beta}=2$.

Exercises L. p. 419.

15. $k(aq-bp)/(a^2+b^2)^{\frac{3}{2}}$. **19.** $\tan^{-1}\left\{\frac{c^2}{2\sqrt{(1-c^2)}}\right\}$.

§ 161, p. 430. **4.** (i) a hyperbola; (ii) a parabola; (iii) a parabola; (iv) an ellipse.

5. $t=-\frac{ab+a'b'}{a^2+a'^2}$.

Exercises LI. p. 432.

1. (i) Ellipse: $a=\frac{1}{2}(\sqrt{10}+\sqrt{2})$, $b=\frac{1}{2}(\sqrt{10}-\sqrt{2})$;
major axis, $(x-1)(\sqrt{5}+1)-2(y-2)=0$;
minor axis, $(x-1)(\sqrt{5}-1)+2(y-2)=0$.

(ii) Hyperbola: $a^2=\frac{8}{5}(\sqrt{2}+1)$, $b^2=\frac{8}{5}(\sqrt{2}-1)$;
transverse axis, $(7+5\sqrt{2})(5x-1)=5y-7$;
conjugate axis, $(7-5\sqrt{2})(5x-1)=5y-7$.

(iii) Ellipse: $a=4$, $b=2\sqrt{2}$; major axis, $x-y=0$; minor axis, $x+y=0$.

(iv) Hyperbola: $a^2=4$, $b^2=3$; transverse axis, $2x+y-1=0$;
conjugate axis, $x-2y-1=0$.

(v) Rectangular Hyperbola: $a=1$, $b=1$;
transverse axis, $4x-3y+10=0$; conjugate axis, $3x+4y-5=0$.

(vi) Parabola: latus rectum $=7\sqrt{10}/100$; axis, $20(3x+y)+9=0$;
tangent at vertex, $40(x-3y)=217$.

(vii) Ellipse: $a^2=45/4$, $b^2=5$; major axis, $x-2y+1=0$; minor axis, $2x+y-3=0$.

(viii) Parabola: latus rectum$=4$; axis, $3x-4y+5=0$; tangent at vertex, $4x+3y-10=0$.

(ix) Hyperbola: $a^2=\sqrt{13}-2$, $b^2=\sqrt{13}+2$; transverse axis, $(5\sqrt{13}-19)(x-3)+2(\sqrt{13}-2)(y-3)=0$; conjugate axis, $(5\sqrt{13}+19)(x-3)+2(\sqrt{13}+2)(y-3)=0$.

3. An ellipse which passes through the origin and touches the lines $x=a$ and $y=b$, where they intersect $x/a+y/b=1$.

4. $ab'+a'b=2hh'$.

5. Directrix: $36(x-y)+77=0$; Focus: $(-23/72, -31/72)$.

7. Latus rectum$=4(a'\sin\alpha-a\cos\alpha)^2$.

Directrix: $(x-b)\sin\alpha+(y-b')\cos\alpha+(a^2+a'^2)=0$.

Tangent at t: $(y-t^2\cos\alpha-2a't-b')=\dfrac{t\cos\alpha+a'}{t\sin\alpha+a}\cdot(x-t^2\sin\alpha-2at-b)$.

9. $(x^2-y^2)/(a-b)=xy/h$.

10. $ax^2+2hxy+by^2+2gx+2fy=0$.

Exercises LII. p. 448.

1. By the Rectangle Theorem, $TP^2/TQ\,.\,T\infty=T'P'^2/T'Q'\,.\,T'\infty$, if T' is a second point on the tangent; therefore $TP^2/TQ=T'P'^2/T'Q'$, and therefore is constant.

2. If the diameter through V, the middle point of PQ, meet the curve in O' and the tangent in T', then

$$\frac{TP^2}{TO}=\frac{T'P^2}{T'O'},\ \text{so that}\ \frac{TO}{T'O'}=\frac{RP^2}{VP^2}=\frac{RP\,.\,RQ}{VP^2}\cdot\frac{RP}{RQ}=\frac{OR}{VO'}\cdot\frac{RP}{RQ}.$$

But $T'O'=O'V$; therefore $TO/OR=PR/RQ$.

12. The point P, where $\angle ASP=\pi/3$.

20. $B=\dfrac{-a^2b}{a^2+b^2}$, $C=\dfrac{-ab^2}{a^2+b^2}$.

26. See § 165.

27. See equation (6) of § 166.

30. $7y^2-24xy+20x=0$.

31. $123x^2+85xy+24y^2=108$.

36. $(x^2+y^2)^2=(a^2+b^2)^2\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\right)$.

40. $\dfrac{e^4}{e^2-1}=\dfrac{(a-b)^2+4h^2}{h^2-ab}$.

INDEX.

The numbers refer to pages.

The numbers refer to pages.

The numbers refer to pages.

The numbers refer to pages.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
THE UNIVERSITY PRESS, GLASGOW.

WORKS ON

CO-ORDINATE GEOMETRY

Works on Practical Mathematics

(Continued)

Manual of Practical Mathematics. By F. Castle. M.I.Mech.E. Globe 8vo. 7s.

A First Book of Practical Mathematics. By T. S. Usherwood, B.Sc., and C. J. A. Trimble, M.A. Globe 8vo. 2s. 6d.

Practical Mathematics for Technical Students. By T. S. Usherwood, B.Sc., and C. J. A. Trimble, M.A. Crown 8vo. Part I. 4s. Part II. 6s. 6d.

Practical Geometry and Graphics for Advanced Students. By J. Harrison, M.I.Mech.E., and G. A. Baxandall. Second Edition. Globe 8vo. 7s.

Mathematical Tables for Ready Reference. By Frank Castle, M.I.Mech.E. Crown 8vo. Paper. 3d.

Logarithmic and Other Tables for Schools. By F. Castle, M.I.Mech.E. 8vo. 9d. Paper. 6d.

Four-Figure Mathematical Tables. By F. Castle, M.I.Mech.E. 8vo. Paper. 1s.

Five-Figure Logarithmic and Other Tables. By F. Castle, M.I.Mech.E. Fcap. 4to. 1s. 3d.